Musical Forces and Two Gauge Models of Tonal Attraction

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Abstract

Metaphors involving motion and forces are a source of inspiration for understanding tonal music and tonal harmonies since ancient times. Starting with the rise of quantum cognition, the interactional conception of forces as developed in modern gauge theory entered the field of theoretical musicology. I argue that the metaphoric conception of musical forces is unable to give a causal explanation of the essential mechanism of tonal dynamics. Rather, it describes correlation without explaining the mechanism that causes these correlations. The gauge-theoretic conception, in contrast, follows a realistic foundation of physical and extra-physical forces. It identifies universal features that are necessary to the unique character and causal function of physical and extra-physical forces. In this squib, we compare two gauge models of tonal attraction. The *phase model* borrows ideas from quantum electrodynamics. It is based on U(1) gauge symmetry. The *spatial deformation model*, in contrast, borrows its main idea from the general theory of relativity and can be seen being based on SO(2) gauge symmetry. In the neutral, force-free case both models agree and generate the same predictions as a simple qubit approach. However, there are several differences in the force-driven case, which we truly debate.

1 Introduction

The phenomenon of tonal attraction has fascinated researchers of music psychology, both empirically and theoretically (Krumhansl & Cuddy, 2010). There is a distinction between two types of tonal attraction, called static and dynamic attraction (Blutner, 2016). How well does a given pitch fit into a tonal scale or tonal key, being either a major or minor key? This is a question of the first type concerning the tonal centers. A question of the second, dynamic type, typically asks for the level of resolution a subject feels when she hears a probe tone following a certain chord in a serial sequence.

In a celebrated study by Krumhansl and Kessler (1982) the static type of tonal attraction was investigated. In this study, listeners were asked to rate how well each note of the chromatic octave fitted with a preceding context, which consisted of short musical sequences in major or minor keys. The results of this experiment clearly show a kind of hierarchy: the tonic pitch received the highest rating, followed by the pitches completing the tonic triad (third and fifth), followed by the remaining scale degrees, and finally followed by the chromatic, non-scale tones. This finding plays an essential role in Lerdahl's and Jackendoff's generative theory of tonal music (Lerdahl & Jackendoff, 1983). It clearly counts as one of the main pillars of the structural approach in music theory. A related approach of the static type is due to Bharucha (1996). The dynamic type of attraction was investigated by Krumhansl (1990, 1995), Lake (1987), Bharucha (1996), Lerdahl (1996), Larson (2004), Larson (2012), and in a recent study of Woolhouse (2009), following earlier research of Brown, Butler, and Jones (1994).

Both types of tonal attraction have not only initiated an enormous number of empirical studies but also challenged a series of different models based on static and dynamic forces. Most of these models are close in inspiration to Larson (2012). All models explicitly or implicitly consider the term "musical forces" as a metaphoric term and build a phenomenological model on this basis. The models that exhibit the metaphorical trait aim to describe correlation. They do not aim describing the causal mechanism underlying tonal attraction.

The work of Mazzola (1990, 2002) is an important exception to this widely shared methodology. His theory sees the whole conception of "musical force" as directly rooted in the basic symmetry principles of tonal music. To distinguish the phenomenological (and metaphoric) idea of forces from the idea based on fundamental symmetries and physical interaction, I will call the latter view the *structural realist view* (abbreviated: *realist view*). The idea is that the force conception plays a causal role in the theory. It is associated with a set of structural attributes that are necessary to the identity and function of physical and extra-physical forces within an interactional, symmetry-based setting.

The present note is concerned with two variants of the realist view, which both explain the existence of musical forces in terms of gauge transformations. The gauge transformation used in the first model, the *spatial deformation model*, is based on vector rotations in a two-dimensional (real) Hilbert space. The model follows ideas that are borrowed from the general theory of relativity. Explicit applications to attraction phenomena were made recently (Blutner, 2016; beim Graben & Blutner, 2017). The gauge transformation used in the second model, the so-called *phase model*, is based on local phase invariances in quantum theory (founded on U(1)). Both gauge models use subgroups of the SU(2) symmetry group by transforming spinors.

The structuring of the paper is as follows. The subsequent Section 2 explains the metaphoric conception of musical forces and contrasts it with a realist conception. Section 3 will introduce the qubit model of tonal attraction. This model can be seen as a special case of the two realist models: it is explicating the force-free (neutral) case. Section 4 explains the general idea of the structural realist view and develops two local gauge theories, which subsequently are applied in music cognition. Both static and dynamic attraction phenomena are discussed. Further, I explain how to approximate the hierarchic model of tonal attraction (Lerdahl 1988, 2001) and how to extend it in order to grasp certain asymmetries (major/minor modes). Section 5, finally, draws some general conclusions and rises several issues for future research.

2 Metaphoric and realist conceptions of musical forces

In classical physics, a force is seen as the cause of any change of the motion of an object. A force has a magnitude and direction making it a vector. According to Newton's second law the force acting upon an object is equal to the rate at which its momentum (= mass times velocity of the object) changes with time. Notably, our intuitive understanding of physical forces is not exactly the same as Newton's physical understanding. This is especially visible in connection with Newton's first law. It states that physical objects continue to move in a state of constant velocity unless acted upon by an external force. This conflicts with our everyday experience assuming that objects move with constant velocity only when a constant force is applied (due to the hidden role of friction or turbulences). Aristotle, to be sure, was much closer to folk physics than Galilei, who was the first who constructed experiments to disprove Aristotle's theory of movement.

Within the last 100 years, the distance between theoretical physics and folk physics has increased even more. In modern particle physics, forces and the acceleration of particles are explained as a mathematical by-product of exchange of momentum-carrying tiny particles (so-called gauge bosons). With the development of quantum field theory and general relativity, it was realized that force is a redundant concept arising from conservation of momentum (4-momentum in relativity and momentum of virtual particles in quantum electrodynamics). The conservation of momentum can be directly derived from the homogeneity or symmetry of space and so is usually considered more fundamental than the concept of a force. Hence, the modern understanding of physical forces sharply contrasts with our folk physical understanding, which is sometimes taken as a sign of progress in science (Weinberg, 1992).

Metaphors involving motion and forces are based on our folk physics and not on the modern understanding of physical forces. The former but not the latter are a source of inspiration for understanding tonal music and tonal harmonies since ancient times. The application of physical metaphors is quite common in theories of music. Physical forces are represented in our naïve (common sense) physics or folk physics. Our experience of *musical motion* is conceptualized in terms of our experience of *physical motion* and their underlying *forces*. For example, Schönberg speaks of different forces when he explains the direction of musical forces in cadences where the tonic attracts the dominant (Schönberg, 1911/1978, p. 58).

Whereas the metaphoric conception is based on analogical reasoning, the realist conception assumes a principle based structural mechanism and deductive reasoning. With the term "realism" I refer to a "group structural realism" (Dawid, 2017). The idea is to distinguish between token-based and type-based realism. In token-based realism (ontological realism), all entities that are postulated within the theory have a direct pendant in reality. In type-based realism, the theory as a whole is tested with reality. It is not required that all postulated entities acquire physical meaning. Perhaps, Johannes Kepler can be seen as a forerunner of the type based realism and the formulation of laws that can be tested only as a whole. Kepler also came with the idea – deeply rooted in his believe in God – that the harmonies of the world and the harmony of music could be described in a uniform way (Kepler, 1619). To be fair, I should add that some authors prefer to interpret Kepler's *Harmonices Mundi* (Kepler, 1619) and his "music of the spheres" in a different way and see Kepler as a forerunner of the metaphoric view (Hubbard, 2017).

In modern physics, gauge theories have provided our best representations of the fundamental forces of nature, including electromagnetic forces, strong and weak nuclear forces. Even when this approach seems not to be reasonable and satisfying in all respects, it is a matter of fact that since more than 50 years local gauge symmetries play an essential role in constructing the most powerful and successful physical theories. A somewhat different mechanism is applied in Einstein's general theory of relativity for describing gravity. In this theory, the configuration space itself (time and space coordinates) is structured by a non-Euclidean metric. A basic assumption is that all coordinate systems (including those that are rotating and accelerated) are equivalent. The force of gravitation relates to a deformation of the configuration space.[1]

2.1 Larson's metaphoric model of tonal forces

Several authors explicitly or implicitly use the ideas of musical movements and musical forces as based on conceptual metaphors in the sense of Lakoff and Johnson (1980). That means the source domain of naïve (folk) physics is assumed to constitute a conceptual network establishing main propositions about physical movements and their causes – the physical forces. Analogical reasoning is used then to transfer the physical concepts to the goal domain of tonal music. In this way, it is possible to describe the most plausible expectations a listener generates during the processing of tonal music. This includes expectations based on static and dynamic forces.

Larson (1997-98, 2004; 2012) is the most prominent author that develops this idea in detail. In particular, he proposed three musical forces that generate melodic completions. These forces are called 'gravity', 'inertia', and 'magnetism', respectively. These forces relate to conceptual metaphors (Lakoff and Johnson 1980) and structure our musical thinking per analogy with falling, inert and attracting physical bodies. Hence, physical forces are represented in our naïve (common sense) physics or folk physics.

Larson (2012) gives some examples that concern ordinary discourses about music. They demonstrate the metaphorical potential of the three forces ('gravity', 'inertia', and

[1] Several attempts were made to provide a gauge theoretic treatment for gravity. Famously, Weyl failed with his first attempt made in 1918 (Weyl, 1950). Later, others were more successful basing the gauge theory on the Lorentz group (1956) or the Poincare´ group (1961).

‘magnetism’). GRAVITY: The soprano's *high* notes rang *above.* The rising melodic line *climbed higher*. MAGNETISM: The music is *drawn* to this stable note. The *leading tone* is *pulled* to the tonic. INERTIA: The accompanimental figure, once set in motion… . This dance rhythm generates such momentum that… (citations at the end of Sect. 8).

In this subsection, I will present the basic ideas of Steve Larson as published in his last book (Larson 2012). I think that this book gives the best overview on the field of musical forces presently available. And it provides a fair discussion on related proposals such as Narmour's (1992) implication-realization model, the model of Bharucha (1996), Lerdahls (2001) algorithm, and related ideas of Margulis (2003) and others.

Larson (2012) investigates the empirical hypothesis that the average rating of each of the investigated patterns is a function of the sum of musical forces acting on that pattern. To do so, a linear regression analysis is performed testing the following hypothesis for the "net force" F for a probe tone x as reflected by the ratings:

(1) $F(x) = w_G \cdot G(x) + w_M \cdot M(x) + w_I \cdot I(x)$

Hereby, w_G, w_M, and w_I are the corresponding weight factors of the three constraint functions. The constraint functions themselves reflect the intuitive content of the phenomenological forces. For instance, the constraint $G(x)$ for gravity gets the value 1 (0) if the probe tone x is lower (higher) than the preceding tone. Hence, the constraint for gravity prefers falling tones to rising ones. The results of the linear regression analysis for the investigated data (Larson & van Handel, 2005) are $w_G = 0.4$, $w_M = 0.1$, $w_I = 1.2$. The correlation between model and data is $r = 0.95$. This high r-value means that the three forces, taken together, can account for about 90% of the variance of the frequency data. The two weight factors for gravity and magnetism are each significantly different from zero (at a 0.1 % level), but the weight for inertia is not. Interestingly, other studies using other data sets (Larson 2002) give a different result: gravity and inertia both make significant contributions but magnetism does not. In the 2005 study an additional analysis was performed that included in addition to GRAVITY, MAGNETISM, and INERTIA, an extra factor signaling *the ending on tonic* (= $\hat{1}$) was introduced. In this case the correlation is still a bit higher: $r = 0.977$, and the extra factor got a weight of 0.46. Interestingly, the other factors now get weights different from the former analysis: $w_G = 0.16$ (instead of 0.4), $w_M = 0.26$ (instead of 0.1), and $w_I = 1.2$ (as before). Hence, magnetism and inertia both make significant contributions to linear regression but gravity does not. This demonstrated that the contribution of single factors to the "net force" can be evaluated only when the full context of all involved factors is given.

Finally, I want to stress that even a high correlation value of the fit as found in the data analysis just described does not answer the fundamental question about constraint grounding. As we have seen, the addition of some extra factors can radically change the influence of other factors and can even marginalize some factors. Hence, a multiple regression analysis with a high overall correlation coefficient cannot be taken as argument that the involved factors are all substantiated and "symbolically grounded" in the sense of Harnad (1990). As such, we cannot expect that these factors play a causal role in explaining tonal attraction.

I think Larson (2012) was aware of these problems. Several of his careful analyses try to justify the special role of musical forces. This contrast with alternative analyses by earlier authors. Further, Larson (2012) has investigated different variants of various factors (constraints) and he found how sensitive the cognitive system reacts even on minimal variations.

2.2 The realist conception of tonal forces

What I will call "realistic conception" in this article is due to ideas borrowed from theoretical physics. According to Penrose (2004), all physical interactions are governed by "gauge connections" which depend crucially on spaces having exact symmetries (p. 289). It are these gauge connections which I will take as the basic of the present realistic conception of tonal forces. From the perspective of quantum physics, the idea of gauge symmetry has been applied by pioneers such as Schrödinger, Klein, Fock and others (for an overview, see Jackson & Okun, 2001).[2] It is suitable to introduce the realist force conception by means of a simplified mechanical picture (following Harlander, 2013).

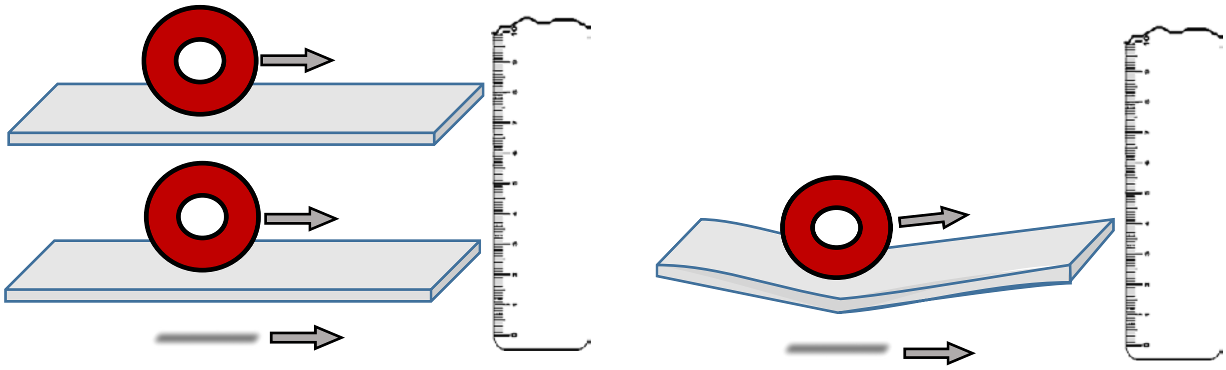

Figure 1: Left: Global Gauge. Right: Local gauge

Figure 1 gives a mechanical example of a so-called gauge symmetry provided by a tire rolling on a pane of glass. The shining sun is producing a moving shadow, which is the essential thing we can observe (similarly to Plato's allegory of the cave). For the movement of the shadow the absolute altitude of the pane is not relevant, only the velocity of the rolling tire is. The fact that the whole scenery of the rolling tire can be moved vertically without changing the movement of the shadow corresponds to a *global symmetry*.

Now assume that there is a deformation of the pane resulting in a local change of the altitude of the tire. The variation of attitude is producing a breaking of the global symmetry. The dynamic effect of the symmetry breaking is that the velocity of the tire is changing by means of the deformation. The shadow at the bottom reflects this behaviour.

The request for *local symmetry* is now simple to understand. It refers to the demand that the movement of the shadow does not give any indication for the deformation of the pane of glass. Obviously, this can happen if we slow down or accelerate the tire dependent on the

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[2] As an example, the description of electrons as formulated by the Dirac equation can be considered. In this case, the multiplication of the wave function with a local phase factor $e^{i\varphi(x,t)}$ introduces an additional term in the transformed Dirac equations which destroys the symmetry. The crucial idea is to compensate the destroying term by an additional term modifying the original electromagnetic potential. This term is seen as describing an interaction of the original electromagnetic field with a gauge field. Obviously, this idea realizes a new dynamical principle coupling the gauge field with the electromagnetic field of the electron. There is a natural interpretation of the gauge field: it describes the interaction of a *photon* with the electron. In other words, the exchange of a photon is realizing a new force found by the idea of a gauge transformation. A more complex case is the standard model of particle physics. The model is formulated as a non-Abelian gauge theory with the symmetry group U(1)×SU(2)×SU(3). It has twelve gauge bosons: the photon, three weak bosons and eight gluons. Between quantum electrodynamics and the full complexity of particle physics, there are symmetry groups such as SU(2) which correspond to the Schrödinger-Pauli equation and U(1)×SU(2) for the Schrödinger-Pauli equation including a Higgs field to give spin-1/2 dyons their masses.

local deformation. In other words, the request of local symmetry demands us to introduce a varying force.[3]

Generally, the idea of founding forces by symmetries is as follows. Assume a physical system is invariant with respect to some global group of continuous transformations (for instance, independence of space and time). Then the idea of gauge invariance, is to make the stronger assumption that the basic physical equations describing the system have to be invariant when the group operations are considered locally (i.e., dependent on time and the other coordinates of the system). Normally, this principle of gauge invariance, leads to a modification of the original equation and introduces additional terms which can be interpreted as new "forces" induced by the "gauge field", which describes these local dependencies.[4]

2.3 The hierarchic model of tonal attraction

Lerdahl (1988, 2001) has developed a model of tonal attraction based on a tonal hierarchy. Forerunners of this approach are Krumhansl (1979), Krumhansl and Kessler (1982) and David Deutsch and Feroe (1981). A numerical representation of Lerdahl's basic space for C-major is given in Table 1. It shows the twelve tones at their levels in the tonal hierarchy. In all, five levels are considered:

A: octave space (defined by the root tone, C in the present case)
B: open fifth space
C: triadic space
D: diatonic space (including all diatonic pitches of C-major in the present case)
E: chromatic space (including all twelve pitch classes).

Table 1 also shows the *tonal attraction* or *anchoring strength s*. This measure simply counts the number of degrees that are commonly shared across levels A to D (omitting level E that is common for all tones).

A:octave	C	x	x	x	x	x	x	x	x	x	x	x
B: fifth	C	x	x	x	x	x	x	G	x	x	x	x
C: triadic	C	x	x	x	E	x	x	G	x	x	x	x
D: diatonic	C	x	D	x	E	F	x	G	x	A	x	B
E: chromatic	C	D♭	D	E♭	E	F	G♭	G	A♭	A	B♭	B
Anchoring strength *s*	0	4	3	4	2	3	4	1	4	3	4	3

Table 1: The basic tonal pitch space as given in Lerdahl (1988).

Temperley (2008) proposes the following formula to calculate the attraction probability:

[3] Of course, pictures such as Figure 1 should be used with great caution. Moses forbid the Israelites to make any image of God. Similarly, in several respects, Dirac remarked that we should not try to make visualizations of quantum theory.

[4] As mentioned already, the idea of gauge invariance was first developed by Hermann Weyl in 1918, when he made the attempt to unify gravity and electromagnetism. Weyl assumed that the length of any single vector is arbitrary. Only the relative lengths of any two vectors and the angle between them are preserved under parallel transport. This was the birth of a new idea in physics which was called "gauge invariance" by Weyl. Even when Weyl's attempt to develop a unified theory failed, the idea survived and was extremely successful later on. It is this success that justifies the theory. The theory itself remains mysterious to a certain degree: we do not have an independent, physical or methodological motivation for it.

$$ (2)\ \ p(j) = \frac{s(j)}{\sum_j s(j)} $$

The predictions of the hierarchical model are in excellent agreement with the experimental data (beim Graben & Blutner, 2018) in the case of major keys. In case of minor keys (based on the natural or harmonic minor scale), however, there are significant deviations (Blutner, 2015; beim Graben & Blutner, 2018).

The basic tonal pitch space is easy to model within the framework of optimality theory (Prince & Smolensky, 1993/2004; Smolensky & Legendre, 2006). In this framework, the tonal levels have to be interpreted by tonal constraints. The constraints simply express whether a given tone is a member of the considered tonal level. For example, the constraint A (related to the octave level) is satisfied if the considered tone is the root tone and it is violated otherwise. In Table 1, a constraint violation is marked by "x". Obviously, the number of violations agrees with the anchoring strength if all constraints are considered equally ranked.

Regarding the *function* of the tonic hierarchy in tonal music, I refer to the insights of Philip Ball, which crucially addresses the tonal dynamics:

> Although it is normally applied only to Western music, the word 'tonal' is appropriate for any music that recognizes a hierarchy that privileges notes to different degrees. That's true of the music of most cultures. In Indian music, the *Sa* note of a *that* scale functions as a tonic. It's not really known whether the modes of ancient Greece were really scales with a tonic centre, but it seems likely that each mode had at least a 'special' note the *mese*, that, by occurring most often in melodies, functioned perceptually as a tonic. This differentiation of notes is a cognitive crutch: it helps us interpret and remember a tune. The notes higher in a hierarchy offer landmarks that anchor the melody, so that we don't just hear it as a string of so many equivalent notes. Music theorists say that notes higher in this hierarchy are more *stable*, by which they mean that they seem less likely to move off somewhere else. Because it is the most stable of all, the tonic is where melodies come to rest. (Ball 2010: 95)

As we have seen, the probe tone techniques used in the experiments by Krumhansl, Kessler and others ask listeners directly to judge how well a single probe tone or chord fits an established context, and the relevant data collected by this technique represent the *static site of tonal attraction*. However, the finding that some tones are more stable than others invites some speculation about the *dynamics of attraction*: When considering sequences of pitches, "a melody is then like a stream of water that seeks the low ground" (Ball 2010: 95). Hence, modifying a picture of Ball (2010), there seem to be forces that are directed toward the tones of the tonic triad (see Fig. 2).

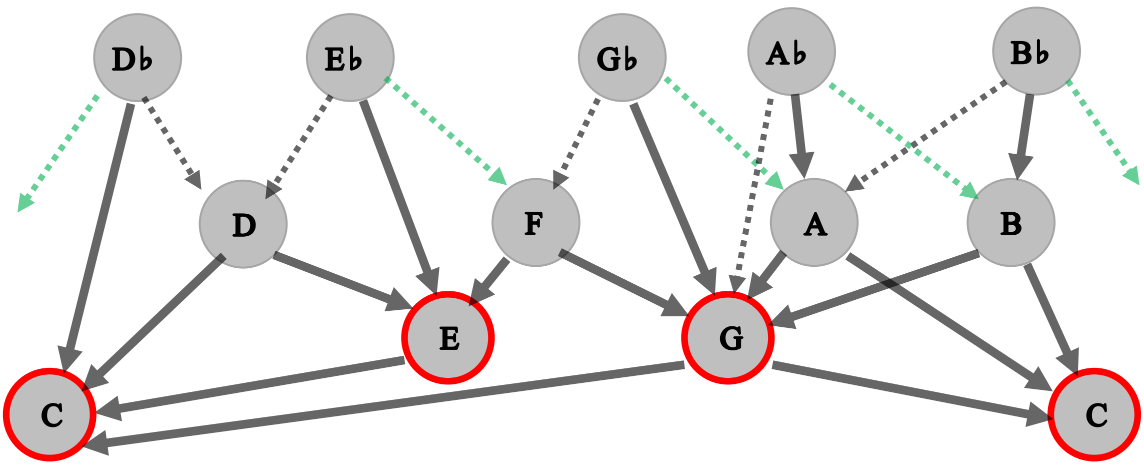

Figure 2: Hypothetical melodic forces (modified from Ball, 2010). The tones of the tonic triad are encircled. Melodic forces are dynamic forces. They are directed towards the chromatically closest tones

that are higher in static attraction than the trigger. The shown example demonstrates the situation for C-major, with the five chromatic tones at the lowest level of attraction.

Importantly, at the state of discussion we have neither a precise mathematical conception of stability nor a realistic idea of musical forces. The metaphoric view of musical forces suggest to take certain pattern of tonal continuation as expression of musical forces. These forces are assumed to determine the dynamics of tonal sequences.[5] This view is based on rather simple-minded ideas about the phenomenology of musical forces. In Sect. 2.1, I have outlined that the underlying multi-regression analysis cannot be taken as a causal explanation of the essential mechanism of tonal dynamics. Rather, it describes correlation without explaining the mechanism that causes these correlations.

In Section 4, I aim to present a causal mechanism to explain the phenomena of static and dynamic attraction. It is based on a realistic view of musical forces and it exploits two alternative views how such forces can arise: (i) in the deformation model based on a local gauge deforming the configuration space; (ii) in the phase model corresponding to a local gauge transforming the phases of the wave function. Remarkably, the realistic view of musical forces does not only give a precise definition of musical forces. It also provides a precise notion of stability, based on the ingenious work of the Russian mathematician Alexander Michailowitsch Lyapunov (Lyapunov, 1966). Before I can introduce the realistic models, it is opportune to explain how basic ideas of quantum cognition can be applied to computational music theory.

3 The qubit model of tonal attraction

For the following, we make use of the notion of a *tonal pitch system*. A tonal pitch system consists of a number of pitches where pitches are sounds defined by a certain fundamental frequency. In this paper, we assume octave-equivalence resulting in twelve pitch classes, also called tones. Further, we assume a tuning system based on an equal temperament, i.e. a tuning system in which the fundamental frequencies between adjacent notes have the same ratio. The following numeric notation is used for defining the twelve tones of the system ("scale degrees" *j*, with *j* running from 0 to 11), in ascending order:

(3) 0 = C, 1 = D♭, 2 = D, 3 = E♭, 4 = E, 5 = F, 6 = G♭, 7 = G, 8 = A♭, 9 = A, 10 = B♭, 11 = B

For applying basic ideas of group theory it is essentially that there are certain operations that allow transforming tones into other tones. For instance, we can increase the tones by a certain number of steps (0, 1, 2, …, 11). Such operations are called *transpositions*. The 1-step transposition transforms C into D♭, D♭ into D, and so on. Operations can be combined. For example, we can combine the transposition of a 2-step increase with a 3-step transposition, resulting in a 5-step transposition (in other words, a major second combined with a minor third gives a fifth). I will denote these operations likewise with the numbers 0, 1, 2, …, 11. Normally, the context makes clear what the numbers denote: a pitch class or the operation of increasing tones by a number of elementary steps. It is obvious that the combination of operations of transpositions can be described by addition (modulo 12): x + y mod 12; e.g., 2+3 mod 12 = 5, 7+6 mod 12 = 1. For a concise introduction of basic concepts of the mathematical theory of groups, the reader is referred to standard text books (e.g., Alexandroff, 2012).

[5] See Sect. 2.1, especially Formula (1). The regression analysis performed by (1) conforms to the dynamics of tonal music. This contrasts with the regression analysis based on the constraints A-D of the tonal pitch space, which clearly corresponds to the phenomenon of static attraction.

In the case of music based on twelve tones, we have to consider the set of group elements {0, 1, 2, …, 11}, and the group operation is $x \cdot y = x + y$ mod 12. The neutral element is the element denoted by 0: $(0 + x)$ mod 12 = $(x + 0)$ mod 12 = x. For the inverse element x^{-1}, we have x^{-1} = (12 – x) mod 12. The group consisting of the 12 tones is a cyclic group, which is called $\mathbb{Z}_{12}$. Note that a group G is called *cyclic* if there exists a single element $g \in$ G such that every element in G can be represented as a composition of g's. The element g is called a *generator* of the group.

In the present numerical representation of the cyclic group $\mathbb{Z}_{12}$ we have four generators conforming to the numbers 1, 11, 7, 5. Hence, 1 (upward) and 11 (downward) generate the sequence of semitones. In addition, the elements 5 and 7 enumerate the group elements in successive fifths or fourths – representing the circle of fifths. Figure 3 gives a visual representation of the group $\mathbb{Z}_{12}$ using the two basically different generators 1 or 11 (left hand side, 7 or 5 (right hand side).

Next, we have to look for a simple geometric representation of this symmetry group. This group could consist of linear maps as studied in linear algebra. More concretely, the group could consist of certain rotations of vectors in a two-dimensional vector space. For instance we can rotate the vector $\psi_\rightarrow = \begin{pmatrix}1\\0\end{pmatrix}$ in 12 steps to the original vector. In linear algebra, the elementary rotation steps can be described by a rotation matrix that is rotating the state vectors by an angle of $\pi/12$.[6] In the Bloch sphere the rotation angle has to be doubled, i.e. $\pi/6$. This is represented on the right hand side of Figure 3 using the generator of the circle of fifth. In contrast, the left hand side shows the 12 tones arranged in a chromatic way. In both parts of Fig. 2, the tones of the diatonic (C major) scale are shown by white circles and the other tones (called the non-diatonic ones) are represented by black circles. Obviously, the 7 diatonic tones as well as the 5 non-diatonic ones are connected (= convex) areas when the *circle of fifth* is used but they are not connected when the *chromatic ordering* is applied. It is this fact that favours the circle of fifth representation over the chromatic representation. The former can be seen cognitively more realistic than the latter (following Gärdenfors' (2000) methodology of conceptual spaces).

[6] Obviously, an explicit representation of the vectors representing the 12 tones in the 2-dimensional Hilbert space is as follows: $\psi_j = \begin{pmatrix}\cos(j\pi/12)\\ \sin(j\pi/12)\end{pmatrix}$ with j = 0, …, 11. For j = 0, we get the tonic vector $\begin{pmatrix}1\\0\end{pmatrix}$ and for j = 6 we get the orthogonal vector $\begin{pmatrix}0\\1\end{pmatrix}$ representing the triton.

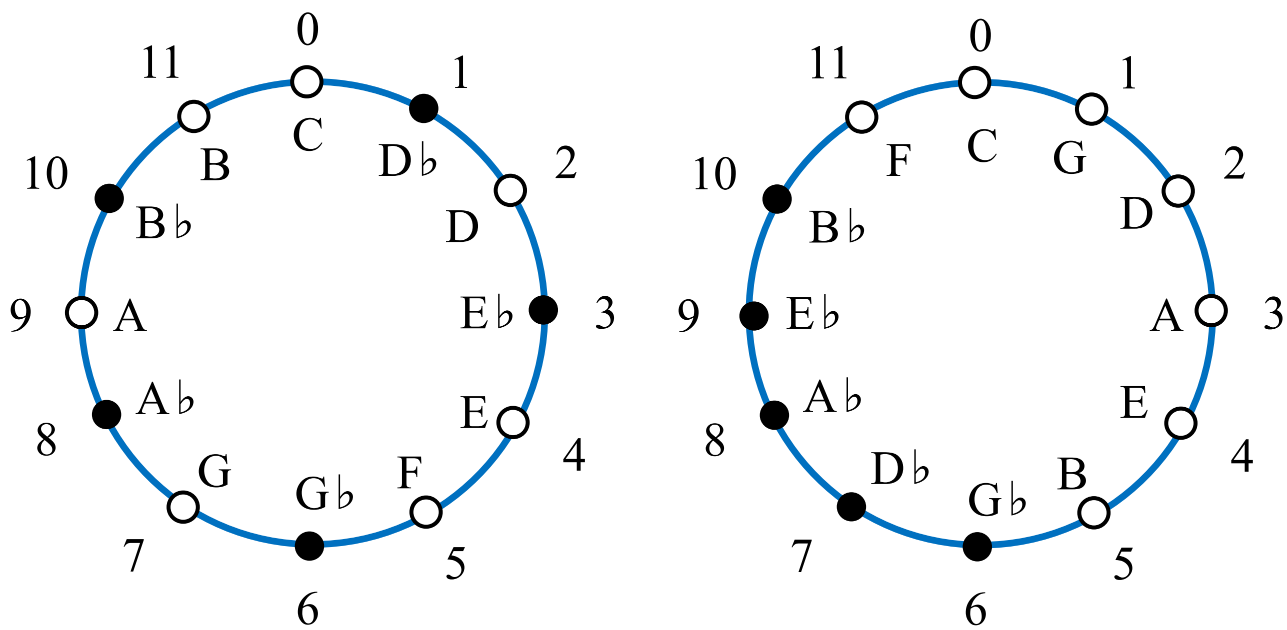

Figure 3: Visual representation of $\mathbb{Z}_{12}$. On the left hand side the different elements of the group are generated by the semi-tone generator. The white dots give an ordered subset of $\mathbb{Z}_{12}$ starting with the tone 0. It is the diatonic scale of C-major. On the right hand side, the group elements are generated by a generator that transposes by seven semi-tones (resulting in the circle of fifth). The numbers indicate how often the generator is applied recursively. The tones in the inner circles are the results of application of the corresponding group element to the basic pitch class C.

There is a straightforward argument for the uniform distribution of the 12 tones in the Bloch circle. It is due to a fundamental symmetry principle. Mathematically, symmetry is simply a set of transformations applied to given structural states such that the transformations preserve the properties of the states. In music, the most basic symmetry principle is the *principle of transposition invariance*. It says that the musical quality of a musical episode is essentially unchanged if it is transposed into a different key, i.e. if the operations of the cyclic group $\mathbb{Z}_{12}$ are applied. Therefore, we can say that $\mathbb{Z}_{12}$ is the symmetry group of (Western) music assuming equal temperament.

In the following subsection, I will explain how this vector representation of the twelve tones makes it possible to derive precise attraction values (in terms of quantum probabilities).

3.1 The neutral case

In the case of pure states, quantum theory defines *structural* probabilities (cf. Blutner 2015). This means the probability that a state ψ collapses into another state depends exclusively on the geometric, structural properties of the considered states. How well does a given tone fit with the tonic pitch of a given tonal context? What is the probability that it collapses into the (tonic) comparison state? The probability $P_l(j)$ of a collapse of the state ψ_j into a state ψ_l (rather than in an orthogonal state) can be calculated straightforwardly. It is the square of the length of the projection of state ψ_j onto state ψ_l:

(4) $P_l(j) = \cos^2(\pi(j-l)/12) = ½(1+\cos(\pi(j-l)/6))$, where $0 \le j, l \le 11$.

For a fixed element ψ_l equation (4) calculates a measure of how well each of the twelve target tones indexed by j ($0 \leq k \leq 11$) fits to the contextually given comparison tone. Hence, formula (4) offers the attraction profile relative to a contextually given cue tone ψ_l. In the following I will set $l = 0$. This allows a simple calculation of the quantum-probabilistic profile assuming a variable k referring to intervals instead of a single tones (the intervals spanned by the contextually cue tone and the target tones):

(5) $P(k) = \cos^2(\pi\, k/12) = ½(1 + \cos(\pi\, k/6))$, where $0 \leq k \leq 11$.

We can compare it with the attraction profile resulting from interval cycles (Woolhouse, 2010; Woolhouse & Cross, 2010), as presented in Fig. 4.

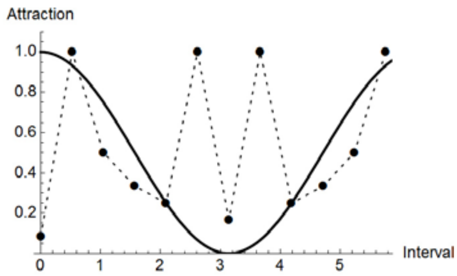

Figure 4: Comparison between the profiles (kernel functions) resulting from interval cycles (dashed) and the profile resulting from a simple quantum model (neutral kernel function; bold curve). Note that the endpoints corresponds to the tonic tone ($0 \cong 12$). Due to the chosen normalization, the maximum of the profiles is 1 in each case.

The figure illustrates that the kernel resulting from interval cycles[7] and the kernel resulting from the simple quantum model are very different.

3.2 The role of phase parameters

When the simple Bloch circle (*real* 2-dimensional Hilbert space) is replaced by the full Bloch-sphere (*complex* 2-dimensional Hilbert space), the phase parameters come into play as shown in the following formula:

(6) $P(k) = \frac{1}{2}(1 + \cos(\Delta_k) \cos(\pi\,(k - 3)/6)$.[8]

[7] The basic idea is that the attraction between two pitches is proportional to the number of times the interval spanned by the two pitches must be multiplied by itself to produce some whole number of octaves. Assuming twelve-tone equal temperament, the *interval-cycle proximity* (ICP) of the interval can be defined as the smallest positive number *ICP* such that the product with the interval length (i.e. the number of half tone steps spanned by the interval) is a multiple of 12 (maximal interval length). For example, the *ICP* for the triton is 2 and the *ICP* for the fifth is 12.

[8] Taking a phase parameter into account, the 12 tones can be represented by the following vectors in the full qubit space: $\psi_j = \begin{pmatrix} \cos(j\pi/12) \\ \sin(j\pi/12)\, e^{i\,\Delta_j} \end{pmatrix}$ with j = 0, …, 11. In case we chose j=0 for the tonic, the calculated projections are identical with the neutral case and do not depend on the phase parameters of the tones. The same holds for the calculated probabilities. Therefore, we take the vector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for representing the tonic. Then it is easy to see that formula (6) results for the probability distribution (attraction potential).

For k= 3, the probability gets its maximum ($P(3) = 1$) if we assume a zero phase shift. In an earlier study (Blutner, 2015) all the phase parameters were fitted by the data of Krumhansl and Kessler (1982). However, it can be criticised that this procedure is not very explanatory because of the big number of parameters that have to be fitted. Further, the status of these parameters as entities that have to be learned is questionable. A more systematic solution is provided in Sect. 4.3 and 4.4 where a gauge-theoretic variant of this model is developed, the *phase model*.

3.3 Attraction profile = kernel + linear convolution

Formula (5) for the neutral case or formula (6) for the complex case with phases are not enough to represent the psychological attraction data as measured by Krumhansl and Kessler (1982) and by others. The point is that these formulas refer to a single contextual element only. In other words, these formulas represent kernel functions only. These kernel function enter a procedure that takes several contextual elements into account (for example, the three tones of a chord or the seven tones of a standard cadence / diatonic scale). The idea we apply here is to handle this procedure by a linear convolution process as represented by the following formula:

(7) $P(j) = \sum_l kernel(j - l) \cdot P_{context}(l)$

You can see this formula as an equation that describes the modification of a kernel function by a distribution of several contextual elements $l \in context$. For example, the kernel function can be the ICP kernel or the function $P(k)$ taken from expression (5) – describing the neutral case of the qubit model. Since the kernel function in (7) depends on the difference $j - l$ only, it automatically satisfies the requirement of transposition invariance.

An important theoretical question is where the distributions $P_{context}(l)$ comes from. The general answer is that it comes from a probabilistic, Bayesian induction, which is based on the frequency of tones realized in a given piece of music. In the simplest case, it is appropriate to take the three tones of a tonic chord – assuming that these three tones realize the context, each of them having probability 1/3. Another possibility is to induce the underlying diatonic scale and to assume that the seven elements of this scale have equal probabilities.

It should be mentioned that Matthew Woolhouse and colleagues were possibly the first who used this methodology for describing attraction potentials (Woolhouse, 2009, 2010; Woolhouse & Cross, 2010) (see also Blutner, 2015). Interestingly, a similar method has been used in computational linguistics for modelling adjectival modification (de Groot, 2013).

4 The realist view of musical forces and two gauge models of tonal attraction

So far, we have considered tones as isolated entities, which are represented by vectors of a two-dimensional Hilbert space. From the point of view of information processing in the cochlea and the anatomy of the auditory brain this is not a very plausible assumption. Already the idea of frequency separation on the basilar membrane suggest a field model of tonal perception, which is closely related to the "place theory" of acoustic processing. The basic idea is the existence of a (one-dimensional) configuration space and the assumption that the different tones relate to discrete parts of this configuration space.

We know that all changes of the quantum state by a unitary transformation leave all observable physical effects unchanged. A unitary transformation does not change the scalar product $\psi_1 \cdot \psi_2$ of two vectors of a Hilbert space. The following matrix describes the general form of a unitary transformation in a two-dimensional Hilbert-space (conforming to the group SU(2)):

(8) $E(\theta,\delta,\tau) = \begin{pmatrix} \cos\theta e^{-i\delta} & -\sin\theta e^{i\tau} \\ \sin\theta e^{-i\tau} & \cos\theta e^{i\delta} \end{pmatrix}$

Hereby, θ, δ, τ are real-valued parameters that determine the transformation. In a field model, these parameters can be independent of the variables that determine the configuration space (i.e. x in the present case) or they can be dependent of these variables (written $\theta(x)$, $\delta(x)$, $\gamma(x)$. In the first case, the transformation is as follows:

(9) $\psi(x) \longrightarrow \tilde{\psi}(x) = [E(\theta,\delta,\tau)]\psi(x)$

It is called *global gauge transformation*. In the second case, the transformation is dependent of the configuration space:

(10) $\psi(x) \longrightarrow \tilde{\psi}(x) = [E(\theta(x),\delta(x),\tau(x))]\psi(x)$

It is called *local gauge transformation*. The invariance of certain equations under global transformations is called *global gauge symmetry* and the invariance under local transformations is called *local gauge symmetry*. The idea behind local gauge transformation is the requirement that this transformation leads to a neutral (force-free) solution of the underlying dynamics. That means the gauge transformation eliminates the forces that control the initial wave function $\psi(x)$, just as in the mechanical example of Sect. 2.2.

Here are two important examples of gauge transformations. First, we consider real-valued vectors and two-dimensional rotation matrices with the rotation parameter θ.

(11) $E(\theta,0,0) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

In the following subsection, we will see that a local variant of this gauge transformation provides the *deformation model* of tonal attraction – a model of static attraction. The advantage of a gauge model of tonal music is that it gives a mathematical derivation of particular musical forces.

Our second example of a gauge transformation is a shift of the phases of the wave function that is illustrated here:

(12) $E(0,\delta,0) = \begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$

The parameter δ is a phase parameter that is assumed being independent of x in the global variant. The local variant of this gauge transformation assumes local dependencies $\delta(x)$ and provides a model of tonal attraction, which is called the *phase model*.[9] Depending on the functions $\delta(x)$ and $\theta(x)$ the predictions of the two models can be rather similar – at least for static tonal attraction. That does not mean that the two models are nearly equivalent. Remarkably, we get important differences when we model dynamic attraction and when we envisage the differences between major and minor modes.

[9] There is a third subgroup of SU(2) describes by the matrix $E(0,0,\tau) = \begin{pmatrix} 0 & e^{i\tau} \\ e^{-i\tau} & 0 \end{pmatrix}$. The transformation triggered by this matrix is fully equivalent to those triggered by the matrix considered in (12).

Obviously, in our simple mechanical picture (see Sect. 2.2), the global attitude of the plate corresponds to the global rotation parameter (in the deformation model) or the global phase parameter (in the phase model). In contrast, the locally changing value of the rotation parameter or of the phase parameter is relevant for the demand of local gauge symmetries. This demand is the structural instrument that introduces physical (and musical) forces. The fundamental equations satisfy the global gauge invariance. The request for local gauge invariance can be satisfied only by introducing additional terms into the basic dynamic equations, which correspond to physical forces. In other words, we have to consider local phase changes always in tandem with emerging forces in the dynamic equations.

There are three main aspects that distinguish the gauge theoretic approaches of tonal attraction from the qubit model of the previous section. First, the qubit model considers tonal states as isolated vectors of a two dimensional vector space (qubit states). It corresponds to the force-free case. In contrast, the gauge theoretic approach analyses tonal states as resulting from a Schrödinger wave function (with its temporal and spatial dimensions). In the simplest case, the wave functions is a standing wave along a one-dimensional spatial continuum $0 \leq x \leq 2\pi$, and the different tones relate to different discrete parts of the configuration space.

Second, the precise shape of the standing wave is determined either by particular local phase shifts (phase model) or by rotations of the vectors dependent of the spatial component (spatial deformation model). In both cases, the twelve tones are described by oscillations of the spin wave at particular points on the spatial axis.

Third, the origin of tonal micro-forces differs for the two models. In case of the phase model it arises from local phase shifts that are described by a parametrized phase function $\delta(x)$. In case of the deformation model it arises from a local rotation function $\theta(x)$ that rotes the spin vectors locally dependent of the spatial x-values. This rotation transformation of the spatial component can be interpreted as a nonlinear transformation of the configuration space and leads to a force conception similar to the idea of the force of gravitation in Einstein's general theory of relativity.[10]

Before I come to the detailed treatment of the two gauge models, I should give a concise introduction into the dynamic aspects of quantum theory. First, I consider the stationary form of the Schrödinger equation for describing objects in (one-dimensional) space and time (Schrödinger, 1926):

(13) $$-\frac{\partial^2 \psi(x)}{\partial x^2} = E\,\psi(x)$$

A solution of this equation under appropriate boundary conditions is the following standing wave:

(14) $$\psi(x) = \cos(\sqrt{E}\cdot x)$$ [11]

For $E = 1/4$ this function is $\cos(x/2)$. It relates to a standing wave along the x-axis with amplitude's maxima at $x = 0$ and $x = 2\pi$ and a zero amplitude at $x = \pi$. Using standard wisdom of quantum mechanics, the probability density for each point of the configuration space is defined by the following expression:

(15) $$|\psi(x)|^2 = \cos^2\left(\frac{x}{2}\right) = 1/2(1+\cos(x)).$$

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[10] It goes without debating that the gauge model is not a modelling of the travelling wave in the cochlea (Terhardt, 1972, 1998). Rather, it could be seen as a third generation neural network approach approaching brain waves in the auditory cortex (Coombes, beim Graben, Potthast, & Wright, 2014).

[11] The solution of the full wave with the stationary part is given by $e^{-i\omega t}\,\psi(x)$

For describing tonal music with 12 tones we need a discretization of the configuration space which we can achieve through sampling $x_k = \frac{\pi k}{6}$, for $k \in \{0, 1, \ldots, 11\}$. The corresponding distribution is then as follows:

(16) $$p_k = \cos^2\left(\frac{\pi k}{12}\right) = 1/2(1 + cos\left(\frac{\pi k}{6}\right))$$

This formula exactly corresponds to the neutral case of the qubit model described by equation (5).

The Schrödinger approach can be straightforwardly extended to the two-dimensional case of objects with spin $\psi(x) = \begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix}$ assuming that the magnetic vector potential does not couple the two spinor components. Hence, I consider the familiar Schrödinger equation for an object in a purely scalar potential, except that it operates on a two-component spinor.

A force-free solution of the Schrödinger equation in this case is $\psi(x) = \begin{pmatrix}\cos(x/2)\\ \sin(x/2)\end{pmatrix}$. For calculating the probability density in the spinor case, we have to assume a projection onto the tonics. If the tonics is the unit vector $\begin{pmatrix}1\\0\end{pmatrix}$, the resulting probability density is exactly as before – see Eq. (15). In case, we take another vector as tonics, let say $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, the result is different:

(17) $$|\psi(x)|^2 = (\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right))^2 = \frac{1}{2}(1 + \sin(x)) = \frac{1}{2}(1 + \cos(x - \frac{\pi}{2}))$$

The difference of $\frac{\pi}{2}$ in the last cosine term of Eq. (17) corresponds to a transposition by three half-tone steps. In the vector representation, the rotation of tonics $\begin{pmatrix}1\\0\end{pmatrix}$ into the new tonic likewise reflects the difference of three half-tone steps. Generalizing this idea, we can prove the principle of transposition invariance of this simple phase model. Things are changing if phase shifts are included into the model (Sect. 4.3).

4.1 The spatial deformation model

In the simplest case, the spatial deformation model (beim Graben & Blutner, 2017) can be described by the following wave function:

(18) $$\psi(x) = \cos(\gamma(x)/2)\,.$$

This wave function yields the following probability density:

(19) $$|\psi(x)|^2 = \cos(\gamma(x)/2)^2 = \frac{1}{2}(1 + \cos(\gamma(x)))$$

We interpret these probability densities as attraction potential for the twelve tones localized at $x_k = \frac{\pi k}{6}$, for $k \in \{0, 1, \ldots, 11\}$

Next, we have to consider a particular choice of the function $\gamma(x)$, which I call *deformation function*:

(20) $$\gamma(x) = a + b\,(x - \pi)^4$$

The deformation factor $\gamma(x)$ increases with the fourth power of the difference between x and the localization of the triton in the Bloch circle (at π). For static attraction we assume that the fixpoints of the gauge transformation are the tonic (corresponding to $x = 0$) and the triton (corresponding to $x = \pi$). This relates to the special parameters $a_s = \pi$ and $b_s = 1/\pi^3$. This assumption reflects a plausible outcome of static attraction experiments that show minimum attraction for the triton and maximum for the tonics.

In the case of dynamic attraction, the triton also should have minimal attraction. In addition, in the dynamic case we can also demand that the tonics gets minimal attraction. This conforms to the parameters $a_d = \pi$ and $b_d = 2/\pi^3$. Figure 5 shows the corresponding kernel functions in the static and in the dynamic cases.

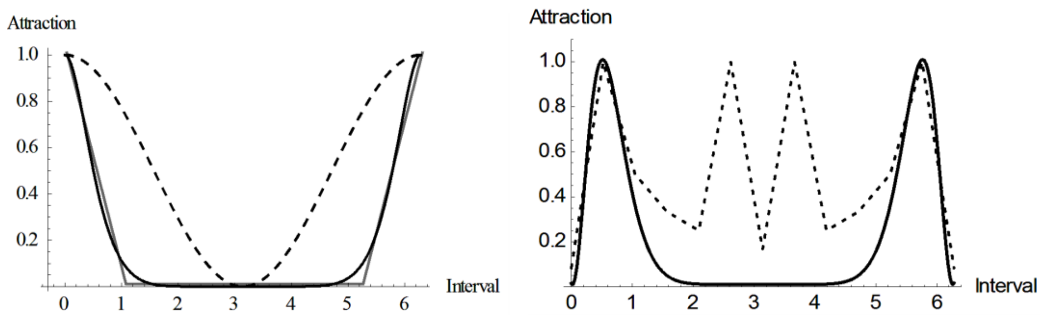

Figure 5: Left hand side: Static kernel (solid) and kernel resulting from the qubit model (dashed). The kernel function resulting from the hierarchic model is shown in grey. Right hand side: dynamic kernel functions (solid) for the spatial deformation model. The kernel functions resulting from the ICM is also shown (dashed). Note that the two endpoints corresponds to the tonic tone.

Earlier research (Beim Graben & Blutner, 2018) has shown excellent agreement with the static attraction data of Krumhansl and Kessler (1982) using the operation of circular convolution (Sect. 3.3) for extending to musical contexts defined by triadic chords. No free parameter has been needed to fit the data (only the exponential in the nonlinear function $\gamma(x)$ can count as a parameter (the exponential 4 leads to fits that are a bit better than other choices such as 2 and 6).[12]

At this point, it is useful to ask for the connection between the deformation model and the hierarchical model. As one sees from Fig. 5, the kernel function of static tonal attraction assigns the maximum value to the target tone (say C). The two neighbours on the circle of fifth (i.e., G and F) get an attraction value that is about half of it. The attraction values of all other tones is very low such that we can neglect them. Hence, when we construct the attraction profiles for a certain context given by a triad (say CEG), we get an approximate reconstruction of the hierarchic model. The three tones of the triad (CEG) get a very high value; C and G a bit higher than E because of the convolution operation. Next, the neighbours of the triadic tones (C: G, F; G: D, C; E: B, A) are all diatonic tones and get an attraction of about 50%. Hence, we can account for all levels of the hierarchic model shown in Tab. 1 besides the octave level (resulting in four different degrees of attraction).[13]

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[12] The treatment of the dynamic attraction function is a bit different from that proposed in beim Graben & Blutner (2018). In this work, a deformation factor $\gamma(x)$ of sixth order has been used. It is determined by achieving a much closer approximations to the ICP model.

[13] At this point readers not educated with quantum mechanics may ask what is the advantage of such complex theories as quantum field theory against the much simpler and classical idea of the hierarchical model? The answer is that the quantum model allows to treat the attraction data with less stipulations than the hierarchical model where all levels of the hierarchy have to be stipulated. Further, the quantum model allows generalizations which are not obvious for the hierarchical model. For instance, the quantum model but not the hierarchical model

4.2 A gauge theoretic formulation of the spatial deformation model

Concerning the dynamic attraction data, I took the ICP model as a guideline for constructing a quantum deformation model of dynamic attraction. The preference of the ICP model for small intervall steps should be met by our model. Furthermore, I claimed that the symmetries of the ICP model should be preserved (cf. Beim Graben & Blutner, 2018). The model is able to describe the dynamic attraction data of Woolhouse (2009) when the corresponding kernel function (see right hand side of Figure 5) has been used. Our model performed comparably well as the ICP model in a regression analysis. Interestingly, our model confirms predictions about the resolution of context chords based on musical harmony theory with good accuracy.

In order to give a gauge-theoretic formulation of the spatial deformation model, we have to replace the scalar wave function given in (18) by a two-dimensional spinor function:

(21) $$\begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix} = \begin{pmatrix}\cos(\gamma(x)/2)\\ \sin(\gamma(x)/2)\end{pmatrix}$$

Now we represent the tonic by the Hilbert-space vector $\begin{pmatrix}1\\0\end{pmatrix}$. Then we straightforwardly obtain an expression for the probability density:

(22) $$|\psi(x)\cdot\begin{pmatrix}1\\0\end{pmatrix}|^2 = \cos^2\left(\frac{\gamma(x)}{2}\right) = \frac{1}{2}(1+\cos(\gamma(x)))$$

This is exactly the result we have derived in Eq. (19). Next, let us consider the real-valued subpart of the SU(2) representation as given in (11). This is exactly the representation of two-dimensional rotations in the real Hilbert space, i.e. SO(2). With the help of this operator, we can rotate the free spinor $\begin{pmatrix}\tilde{\psi}_+(x)\\ \tilde{\psi}_-(x)\end{pmatrix} = \begin{pmatrix}\cos(x/2)\\ \sin(x/2)\end{pmatrix}$ into the spinor (21) deformed by gauge forces (cf. Beim Graben & Blutner, 2018):

(23) $$\left[E\left(\frac{1}{2}(\gamma(x)-x),0,0\right)\right]\begin{pmatrix}\cos(x/2)\\ \sin(x/2)\end{pmatrix} = \begin{pmatrix}\cos(\gamma(x)/2)\\ \sin(\gamma(x)/2)\end{pmatrix}$$

Intuitively, tones x can be described by certain positions in the Bloch circle (angle x). The gauge transformation rotates the corresponding vectors by an angle $\gamma(x) - x$ in the Bloch circle.[14] The addition theorems of cosine/sin-functions give the rotated vectors as exhibited in Eq. (23). Figure 6 illustrates the gauge transformation depictured in the circle of fifth. The twelve tones, which are uniformly distributed over the circle and localized at $x_k = \frac{\pi k}{6}$, for $k \in \{0, 1, \ldots, 11\}$, are mapped by the gauge transformation into a deformed distribution.

relates static and dynamic attraction phenomena. Further, the quantum model but not the hierarchical model allows to approximate the differences between major and minor modes. The situation is similar to the situation in physic more than 100 years ago where the spectrum of the hydrogen atoms has been investigated and described by the simple Rydberg formula (Rydberg, 1890). The derivation of this formula by Bohr's atomic model and later by quantum theory counts as a milestone of physical progress, because of its explanatory value and numerous possible generalisation in the field of atomic spectra.

[14] In order to avoid confusions: The variable x measures the angles in the Bloch-circle. It runs from 0 to 2π. The 'real' angles of the Hilbert-space vectors are half of it. Thus, in the 'real' rotation matrix (23) we have to multiply the angles of the Bloch circle by the factor ½ in order to get the angles in the 'real' rotation matrix.

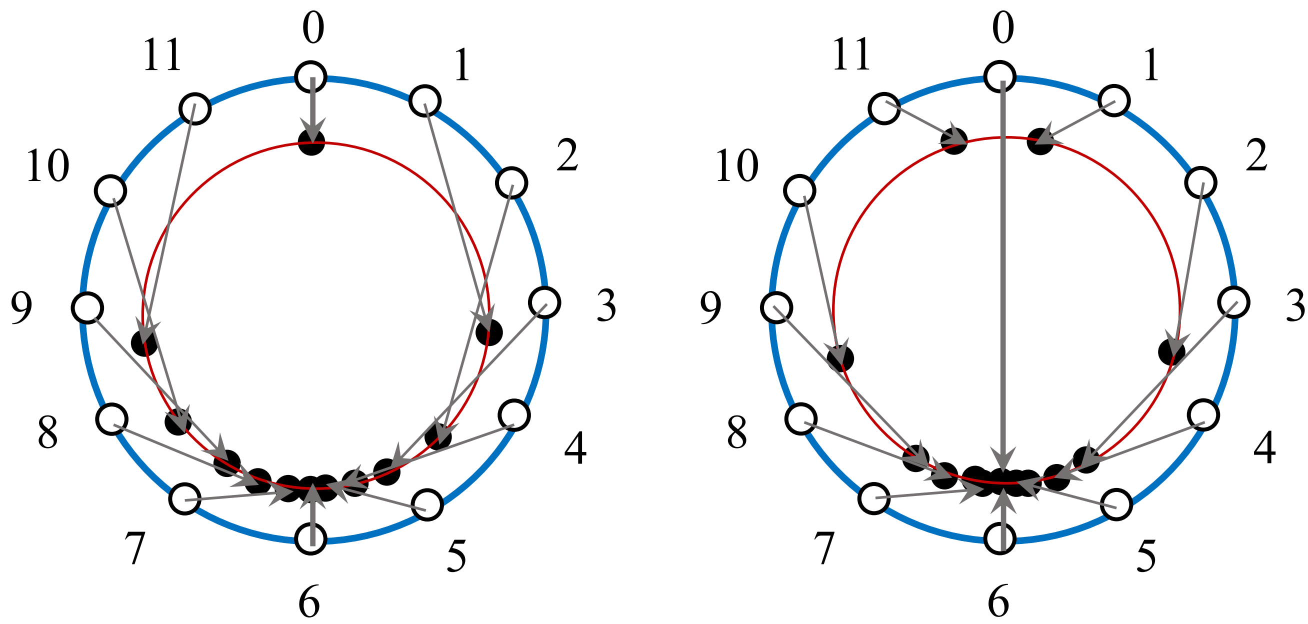

Figure 6: Illustration of SO(2) gauge transformation for the twelve tones (originally located at $x_k = \frac{\pi k}{6}$, for $k \in \{0, 1, \ldots, 11\}$ in the Bloch circle). Left hand side: static gauge function $\gamma_s(x) = \pi + 1/\pi^3\,(x-\pi)^4$. Right hand side: dynamic gauge function $\gamma_d(x) = \pi + 2/\pi^3\,(x-\pi)^4$

Figure 6 illustrates that all tones with exception of the tonics itself and the triton are moved away from the tonics into the direction of the triton. Intuitively, it may be helpful to conceptualize this by 'gauge forces' that cause the deformation depictured in this figure.

Now consider a local gauge transformation, which converts the force-free solution of the Schrödinger equation into the deformed solution under the influence of gauge forces when we assume that $\gamma(x) = \theta(x) - x$:

(24) $$\begin{pmatrix}\tilde{\psi}_+(x)\\ \tilde{\psi}_-(x)\end{pmatrix} \rightarrow \begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix} = \begin{pmatrix}\cos\theta(x) & -\sin\theta(x)\\ \sin\theta(x) & \cos\theta(X)\end{pmatrix}\begin{pmatrix}\tilde{\psi}_+(x)\\ \tilde{\psi}_-(x)\end{pmatrix} = \begin{pmatrix}\cos(\gamma(x)/2)\\ \sin(\gamma(x)/2)\end{pmatrix}.$$

Let us see the correspondence to the free Schrödinger equation (13) in case the wave function $\psi(x) = \tilde{\psi}(\gamma(x))$ is considered. As argued in detail in another paper (beim Graben & Blutner 2017, 2018), the function ψ has to satisfy the following equation for both components of the spinor:

(25) $$-\frac{\partial^2\psi_i(x)}{\partial x^2} + \frac{\gamma''(x)}{\gamma'(x)}\frac{\partial\psi_i(x)}{\partial x} - \gamma'(x)^2\,\psi_i(x) = \mathrm{E}\,\psi_i(x)$$

This equation is derived by differentiating $\psi(x)$ twice and eliminating trigonometric terms. In order to get the standard stationary form of the Schrödinger equation, I decompose the Hamiltonian into three parts:

(26) $H = T + M + U$ with

a. $T = -\frac{\partial^2}{\partial x^2}$

b. $M = \frac{\gamma''(x)}{\gamma'(x)}\frac{\partial}{\partial x}$

c. $U = E - \gamma'(x)^2$.

Then Eq. (25) takes the standard form of an eigenvalue problem:

(27) $[T + M + U]\, \psi_i(x) = E\, \psi_i(x).$

It is obvious to call the operator *T* the operator of kinetic energy and to call the operator *U* the operator of potential energy. Depending of the special form of the gauge field $\theta(x)$ (or $\gamma(x)$) the kind of potential energy that is involved is alike a 'gravity' potential, a 'harmonic oscillator' potential or what else. The operator *M* has been called magnetism in analogy to the physical examples. The details of gauge function $\gamma(x)$) specify the magnetism function.

It is mainly the sum of the three energies, which really has a musical interpretation. The decomposition into the three parts currently cannot be related with any musical phenomenon. To be sure, this decomposition has absolute nothing to do with forms of the metaphoric model that use different contributions of "folk-physical" forces in a linear regression analysis of tonal attraction (e.g. Larson 2012).

I also should stress the positive outcomes of the present gauge analysis. In contrast to the mentioned analyses by linear regression, the gauge-theoretic analysis is not ad hoc and does not require many arbitrary stipulations. The only assumption I have to make is to make a choice of the gauge field $\gamma(x)$. I have decided to approximate the gauge field by the function (20) in order to express the benefits of the hierarchical model. However, the present formulation is more than a sophisticated conversion of the hierarchic model into an oversized mathematical model. Conceptually, it is a direct and realistic introduction of musical forces based on the lead of mainstream physical ideas. Empirically, I will show how it helps to relate static and dynamic attraction models. Further, I will demonstrate that a modification of the model by exploiting the idea of symmetry breaking can helps to solve the old problem of the harmonic differences between major and minor modes of tonal music.

I mentioned already that the sum of the three operators gives an energy density that is proportional to the probability density. A further plausible assumption is that the potential energy is an indicator of stability – cf. Graben & Blutner (2017) and the remark at the end of this subsection. In the following figure, I compare the overall energy density with the contribution resulting from the potential energy density (summing up *M* and *U*).

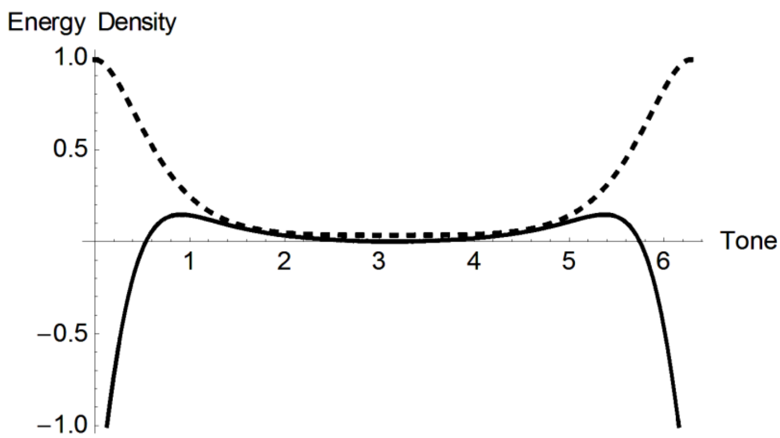

Figure 7: Energy densities of the deformation model. Dashed: total energy density; solid: density of potential energies (*M*+*U*).

As pointed out in more details in beim Graben & Blutner (2017), the overall energy density (proportional to the static attraction potential) has a maximum at the tonic tone (localized at 0 and 2 π in Figure 7) and a minimum for the tritone Localized at π). The sum of the potential energies becomes minimal toward the tonic regions. This explains the attracting force of the tonic for all other tones. Interestingly, there is a local minimum at the localization of the triton, making the triton to a kind of 'tonal trap' for tones of the region D, A …E♭, B♭ (corresponding to the interval from .8 to 5.4). Further, Figure 7 shows that there are two instable equilibria around x = .8 (D) and x= 5.4 (B♭).

At this point, I should explain the content of the mathematical conception of stability. In quantum theory, there is the distinction between *energetic stability of matter* and *dynamic stability of motion*. The former notion refers to the stability of atoms or macroscopic matter. Usually it is explained by the uncertainty relation (in a particular form) and by Pauli's exclusion principle (Lieb & Seiringer, 2010). The latter notion derives from ideas of Lyapunov and the existence of the so-called Lyapunov function the (local) minima of which describe to stable configuration of a dynamic system (Lyapunov, 1966). In the context of cognitive musicology, the stability of tonal movements is a point of interests – clearly referring to the concept of dynamic stability. Understandably, the potential energy of a system is such a Lyapunov function. And the gradients of the energy density correspond to the mathematical notion of forces, both in mechanics as in musicology. They indicate the direction of physical or tonal movements. Summarizing, the present analysis indicates that tonic appears as a center of force. In contrast, the tritone functions as a tonal trap attracting nine of the twelve tones with a moderate force.

4.3 Naïve gauge models based on phase shifts

The general matric form of a gauge transformation for spinors has been given in (8). A subspecies of this gauge was considered in the previous section, the SO(2) gauge. In this and the following Section we consider gauges isomorph to U(1) as specified by (12) for the spinor case. In the simpler case of scalar wave functions, this gauge relates to a single phase transformations. In the local case, we assume that all observable effects are invariant when *local* phase transformations are applied:

(28) $\psi(x) \longrightarrow \tilde{\psi}(x) = \psi(x)\, e^{i\delta(x)}$

The idea now is to assume that local phase invariance is the gauge symmetry that we have to assume in quantum physics.

Introducing a local phase shift as defined by Eq. (28) and the idea of gauge invariance automatically converts the force-free solution of the Schrödinger equation into a solution under the influence of gauge forces. To see the idea, let us consider the following free wave function $\cos(x)$ which is gauged by the local phase transformation $e^{-i\delta(x)}$.

(29) $\psi(x) = \cos(x/2)\, e^{-i\delta(x)}$

A simple consequence of this choice is the following function of probability density:

(30) $\psi(x)^* \cdot \psi(x) = \cos(x/2)^2 = 1/2(1+\cos(x))$

Obviously, the probability density does not depend on the gauge function $\delta(x)$.

For performing a gauge analysis, let is differentiate the function $\psi(x)$ twice and eliminating trigonometric terms. Then the function $\psi(x)$ is a solution of the following differential equation:

(31) $\quad -\psi''(x) - 2i\,\delta'(x)\psi'(x) - i\,\delta''(x)\psi(x) + (E + \delta'(x)^2)\,\psi(x) = \mathrm{E}\,\psi(x).$

Considering the stationary Schrödinger equation in the form (13) suggests a gauged Hamiltonian, which consists of a sum of three operators:

(32) $\quad \breve{H} = \breve{T} + \breve{M} + \breve{U}$ with

a. $\breve{T} = -\frac{\partial^2}{\partial x^2}$
b. $\breve{M} = -2i \cdot \delta'(x)\frac{\partial}{\partial x}$
c. $\breve{U} = E + \delta'(x)^2 - i\delta''(x).$

As in the spatial deformation model discussed in the previous section, it is obvious that the operator $\breve{T}$ is the operator of kinetic energy density (inertia) and the operator $\breve{U}$ is the potential energy density. Physicists call the operator $\breve{M}$ *magnetic potential*. Taking the wave function (29) for calculating the expectation values, the energy densities contain imaginary parts. The sum $\breve{H}$ of the three operators, however, is Hermitian and has the real expectation value $1/4\cos(x/2)^2$.

For specifying the gauge function[15], a plausible suggestion is the following nonlinear ansatz with a quadratic term and two positive valued parameters $\breve{a}$ and $\breve{b}$:

(33) $\quad \delta(x) = \breve{a} + \breve{b}(x-\pi)^2.$

This function has a minimum phase shift for $x = \pi$ and it is monotonic increasing in both directions of the x-axis. According to Eq. (32)c, this describes a potential proportional to $-(x-\pi)^2$ with decreasing values increasing distance from the triton (in direction to the two endpoints 0 and 2 π). Mechanically, this corresponds to the situation of a moving coin that can roll down a hill into two directions with increasing values of slope. Assuming a zero phase shift for the triton ($x = \pi$) and a phase shift of π for the tonic, we can fix the parameters by $\breve{a} = 0$ and $\breve{b} = 1/\pi$. In the present case, the total energy density is ¼ of the probability density, which is $\cos(x/2)^2$ and does not depend on the phase function. However, the density of the potential energy depends on the phase function as shown by Formula (32)c. The expected total potential density (*M* + *U*) is the product of a form factor ($= E - \delta'(x)^2$) and the probability density. Figure 8 shows the expected densities of potential energy (*M* + *U*) and total energy (*T*+*M* + *U*) for the stationary wave function $e^{-i\delta(x)}\cos(x/2)$. Note that the total energy density function is simply proportional to the probability density.

[15] The function $\delta(x)$ is our *gauge function* – it realizes a definite the *gauge field* (corresponding to its first derivation, i.e. $\delta'(x)$).

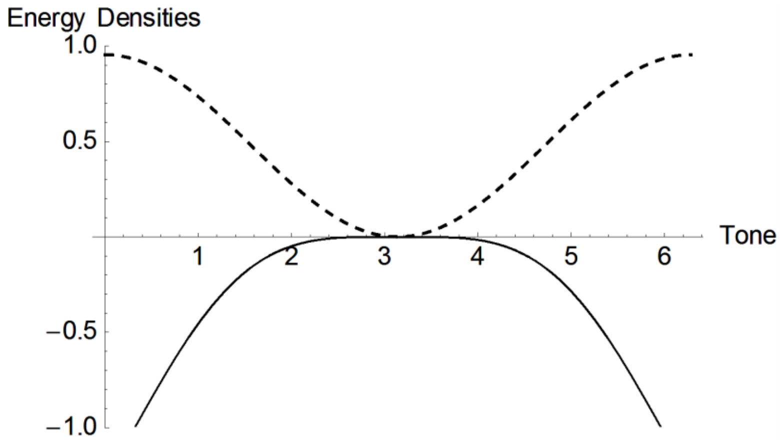

Figure 8: Phase model with stationary wave function $\psi(x) = \cos(\frac{x}{2})e^{-i\delta(x)}$. Dashed: total energy density; solid: density of potential energies (without the imaginary part of magnetism).

What are the main properties of the calculated energy densities? First, it is obvious that the potential energy density has the maximum for the triton and the minimum for the tonic tone. The converse holds for the total energy density, which is proportional to the probability density. Hence, the tonic tone is the stable endpoint. It has maximum probability density and minimum potential density. On the other hand, the triton is in an instable equilibrium state. Presently, it is unclear, whether a musical interpretation of this property can be given (possibly in connection with the triton paradox? – cf. Deutsch, 1991). All other tones are instable and are affected by musical forces. In the present framework, musical force are nothing else than the gradient of the potential energy density. This force is proportional to $x - \pi$. Hence this force has the largest negative value close to the lower tonic and the largest positive value close to the upper tonic. Negative force pulls towards smaller x-values and positive force pulls toward larger x-values. Since the force field is monotonic increasing, it comes out that C attracts G, G attracts D, D attracts A, and so one. Towards the other endpoint, B♭ attracts E♭, F attracts B♭ and C attracts F. Of course, this outcomes hold for contexts that are established by a single note only, not for contexts consisting of chords, cadences or whole scales. In this case, we could assume the additivity of forces and calculate the resulting force.

We have seen already that introducing a local phase shift into the wave function does not change the resulting probability distribution. It still agrees with the results of the simple qubit model discussed before. A totally different solution results from the spinor model, which I will consider in the next subsection. For the spinor's gauge model, I also will consider the additivity of energies (and forces) and I will calculate the averaged potential.

4.4 Spinor gauge models based on phase shifts

In this subsection, I will look for the force-free wave equation in case of spin ½ particles. In this case, we can apply the so-called Schrödinger-Pauli equation, which sometimes has been seen as a special case of the Dirac equation in the non-relativistic limit. The following formula gives a (minimalist) solutions of the Schrödinger-Pauli equation in the force-free case (cf. Blutner 2016):

(34) $\psi(x) = \begin{pmatrix} cos(x/2) \\ sin(x/2) \end{pmatrix}$

For a gauge-theoretic analysis, we have to consider the matrix form of the SU(2) group as given in (8). It provides a unitary transformation of spinors with the real parameters θ, δ, τ.

If we want to simulate the properties of the tones along the circle of fifth, we have to find a proper path through this space of parameters. For the present phase model, we consider $\delta(x)$ as the only relevant parameter and chose $\theta(x) = 0, \tau(x) = 0$. Applying the corresponding unitary transformation $E(0, \delta(x), 0) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\delta(x)} & 0 \\ 0 & e^{i\delta(x)} \end{pmatrix}$ to the spinor's wave function (34), we get the following transformed wave function (for details and motivation see Appendix 2):

(35) $\psi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} cos(x/2)e^{-i\delta(x)} \\ sin(x/2)e^{i\delta(x)} \end{pmatrix}$

In quantum field theory, the projections of the field vectors on certain states provides the corresponding probability densities. In the simplest case, we project onto the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This corresponds to the projection operator $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and we can calculate the probability density relative to this operator by calculating its expectation value for our stationary wave function (35). The result is identical with the scalar case (30) considered before.

However, inspirited by the qubit model with phase factors (Sect. 3.2), we can introduce another tonic operator $\mathbb{T}$, which describes the projection onto the tonic vector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

(36) $\mathbb{T} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

With the help of this operator, the probability density can be calculated as follows:

(37) $\psi(x)^* \cdot \left(\mathbb{T}\,\psi(x)\right) = \frac{1}{2}(1 + \cos 2\delta(x) \cos(x - \frac{\pi}{2}))$

Taking the series of points $x_k = \frac{\pi(k-3)}{6} = \frac{\pi\, k}{6} - \frac{\pi}{2}$, the following probability distribution will result:

(38) $P(k) = \frac{1}{2}(1 + \cos\left(\breve{\delta}(x_k)\right) \cos\left(\pi\ (k-3)/6\right))$, with $\breve{\delta} = 2\delta$.

This is exactly the earlier distribution (6) of the qubit model with phase parameters. Again, the tonic is represented by $k = 3$. Hence, the spinor model produces the same probability distribution as the qubit model considering the phase parameters.

At the deformation model, the spinor phase model gives a simple possibility for calculating the static and dynamic attraction potentials. Further, it equally allows the introduction of gauge forces – for example by using a simple ansatz for the phase function $\delta(x)$ as in eq. (33). Unfortunately, we cannot fix the parameters $\breve{a}$ and $\breve{b}$ in the static case by simply assuming that the probability density given by (38) should have the maximum value 1 for the tonic and the minimum value for the triton. There are many solutions under these conditions.

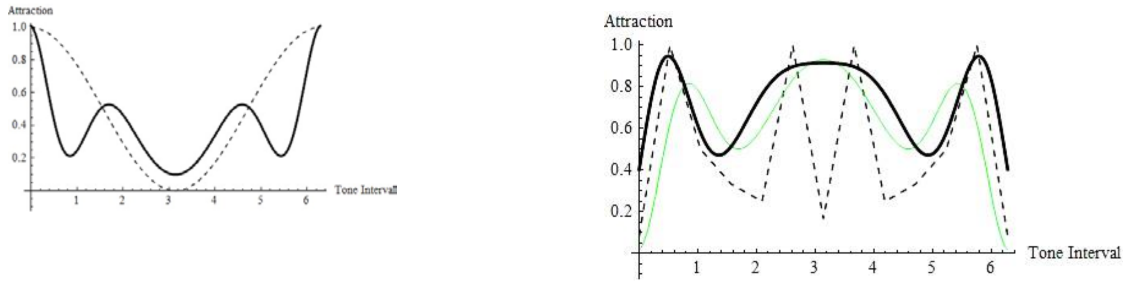

Figure 9: Left hand side: Static kernel function involving phase shift (solid) and static kernel for free wave (dashed). Right hand side: dynamic kernel functions for the phase model (solid) contrasted with the ICP (dashed). The fitted parameter is $\breve{\alpha} = 2.52$. This corresponds to a phase shift of about 2 between static and dynamic case. The thin curve shows a dynamic kernel with phase shift π. In this case, the static and dynamic kernels are exactly mirror-symmetric relative to an attraction value of ½.

However, we can fit the two parameters by Krumhansl's & Kessler's (1982) data for C-major. In the static case, this the following values: $\breve{\alpha} = 0.588$; $\breve{b} = 0.565$. Figure 9 shows the corresponding result (with tonic at 0 and 2 π and triton at $\pi -$ using the new variable $y = x - \frac{\pi}{2}$). For treating the dynamic case, we will assume that the two kernels are nearly mirror-symmetric relative to an attraction value of ½. We get this configuration by increasing the first parameter by π. Fitting the Woolhouse data in the dynamic case, instead gives the fitted value $\breve{\alpha} = 2.52$. Hence, the fit of this parameter results in a global phase difference of about 1.64 between dynamic and static case. Table 3 compares static and dynamic attraction for the deformation model and the phase model. For the deformation model, we have used the results of beim Graben & Blutner (2018). In brackets, the correlation values with the dynamic kernel function shown in Fig 5 (right hand side) are shown.

Empirical Data		Deformation Model (correlations with data)	Phase Model (correlations with data)
Static Attraction	major	.97	.855
	minor	.95	.85
Dynamic Attraction	C-major	.44 (.37)	.28
	C-minor	.93 (.47)	.70
	dominant seventh	.65 (.66)	.59
	French sixth	.76 (.85)	.96
	half-diminished seventh	.90 (.53)	.86

Table 3: Comparing static and dynamic attraction for the deformation model and the phase model. The static attraction data are from Krumhansl & Kessler (1982). The dynamic attraction data (Woolhouse 2009) concern (a) major triad CEG, (b) minor triad CE♭G, (c) dominant seventh CEGB♭, (d) French sixth CEG♭B♭, and half-diminished seventh CE♭G♭B♭.

In case of static attraction, the deformation model appears to fit Krumhansl's & Kessler's (1982) data a bit better than the phase model (even when the differences are not significantly different at a level smaller than 5 %). For dynamic attraction, the comparison between deformation model and phase model does not lead to results that are more conclusive. From a conceptual point of view, the phase model might have the advantage, that only *one* parameter is needed to approximate the dynamic data: the global phase shift between dynamic and static phase kernel function.

Next, let us consider the energy densities in the spinor model. We have to use the following operators:

(39) $\hat{H} = \hat{T} + \hat{M} + \hat{U}$ with

a. $\hat{T} = -\frac{\partial^2}{\partial x^2}\mathbb{T}$

b. $\hat{R} = -2i \cdot \delta'(x)\frac{\partial}{\partial x}\sigma_3\mathbb{T}$

c. $\left(\hat{U} = E + \delta'(x)^2 - \mathrm{i}\,\delta''(x)\right)\mathbb{T}$.[16]

Hence, instead of the scalar operators $\breve{T}, \breve{M}, \breve{U}$ of Sect. 4.2 we now have to consider the corresponding operators on spinors (labelled $\hat{T}, \hat{M}, \hat{U}$). For the definition of these operators, we make use of the tonics operator $\mathbb{T}$ of Eq. (36). The inclusion of this operator ensures that all probabilities and all expectation values are calculate relative to the tonics. Note further that the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is involved.

An explicit calculation of the energy densities of these operators based on the phase shift function (33) is straightforward. Figure 10 shows the total energy density and the density of potential energies (left hand side).

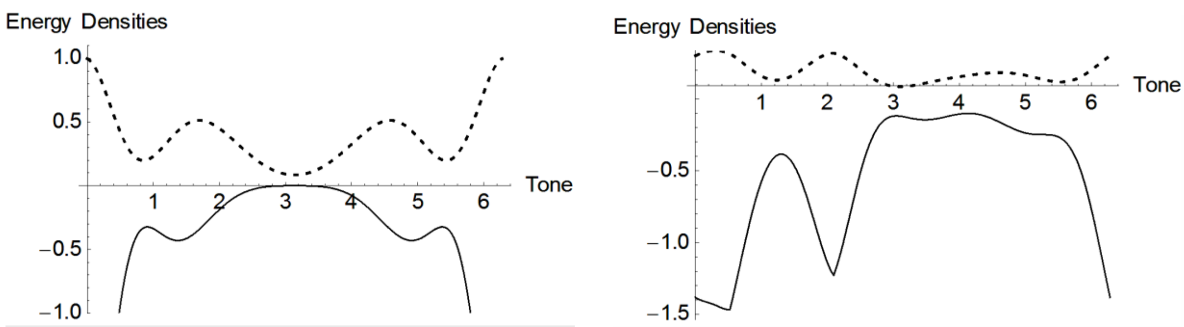

Figure 10: Phase model with stationary wave function as given in (35) in the static case. Dashed: total energy density; solid: density of potential energies. Left hand side: kernels; right hand side: averaging for C-major triad.

As in the cases considered before, the total energy density corresponds with the probability distribution of the (static) attraction function, which I have discussed already. The same holds for dynamic attraction by using the globally changed phase factor (not shown).

If we compare the density of potential energy for the phase model with the density of potential energies for the deformation model, we find agreement about the fact that the tonic (localized at 0 and 2π) corresponds to a global minimum (stability). However, we also find two important differences: (i) In the phase model, the triton is not trapped as in the deformation model. (ii) In the phase model but not in the deformation model we find two additional minima of the potential energies, which are clear indicators for stability. These minima are close to the tones D, A (and B♭, E♭) assuming the tonic is the single tone C.

On the right hand side of Figure 10 we have averaged over the tonic triad CEG of C-major. The local minima are close to the tones C, E, G. This demonstrates the obvious fact that the chosen key in the static case should be stable. Interestingly, when we take the parameters for the dynamic attraction curve, we get three other local minima conforming to the tones C, A, and F (forming the sub-dominant triad for C-major) – an outcome that is not implausible.

[16] Note that simply $\hat{H} = \breve{H}\,\mathbb{T}$, $\hat{T} = \breve{T}\,\mathbb{T}$, $\hat{M} = \breve{M}\,\sigma_3\mathbb{T}$, and $\hat{U} = \breve{U}\,\mathbb{T}$.

4.5 The hierarchical model and the connection between static and dynamic attraction

It is useful to ask for the connection between the deformation model and the hierarchical model. As you see in Figure 5, the kernel function of (static) tonal attraction assigns the maximum value to the target tone (say C). The two neighbours on the circle of fifth (i.e., G and F) get an attraction value that is about half of it. The attraction values of all other tones is very low such that we can neglect them. Hence, when we construct the attraction profiles for a certain context given by a triad (say CEG), we get an approximate reconstruction of the hierarchic model. The three tones of the triad (CEG) get a very high value; C and G a bit higher than E because of the convolution operation. Next, the neighbours of the triadic tones (C→G,F vs. G→D,C vs. E→B,A) are all diatonic tones and get an attraction of about 50%. Hence, we can account for all levels of the hierarchic model shown in Table 1 with exception of the octave level. Concluding, the instrument of analytical functions – by using a strongly damped function – provides the mathematical instrument for describing the hierarchical model.

As outlined in Sect. 2.3, Philip Ball has proposed a connection between static and dynamic attraction (Ball 2010). He suggested a simple, tentative principle saying that certain melodic forces are dynamic forces that are directed towards the chromatically closest tones that are higher in the static attraction hierarchy than the trigger (let us call it *Ball's principle*). Figure 2 (Sect. 2.3) illustrates this idea. The present gauge models establishes a close connection between static and dynamic attraction. Hence, we can expect that it makes predictions that are similar to those conforming to Ball's (2010) expectations. The comparison is illustrated in Table 4. We consider C-major as the underlying scale and we consider all chromatic ('out-of-scale') trigger tones. The table shows the correlation between Ball's (1990) expectations[17] and the predictions of the ICP model, which gives similar results as the two variants of the gauge model for dynamic attraction.

Triad Class C-Major	*Target tone*	*Correlation. Ball - ICP*	*Ball's (2010) principle*
	D♭	.25	**C** *B D*
	E♭	.42	**E** *D F*
	G♭	.43	**G** *F A*
	A♭	.53	**A** *G B*
	B♭	.50	**B** *A C*

Table 4: Dynamic attraction profiles for the five chromatic triggers in a C-major context. The correlations between Ball's principle and ICP are listed. On the right hand column, Ball's principle is exemplified. The tone with the highest predicted attraction is listed in bold together with the next two tones in the attraction hierarchy (in Italics).

Since Ball's principle is not empirically validated, so far we can see, we have to restrict ourselves to a theoretical comparison. Table 4 shows that there are small positive correlations between the predictions of Ball's principle and those made by the ICP model. Similar predictions are made when we use the present variants of the gauge model applied to dynamic attraction.

[17] To make an explicit calculation possible, we have quantified Ball's Predictions. The value 2 is assigned to the probe tones with strongest attractions (bold arrows in Figure 2), the value 1 is assigned to the probe tones of lower level of attraction (dashed arrows) and all other probe tones get the value 0.

4.6 The breaking of the mirror symmetry of the hierarchical model

In the preceding sections, we have considered the conception of static and dynamic attraction only. However, in cognitive music theory some other basic conceptions have been discussed including the idea of graded consonance/dissonance. According to Parncutt (1989), the degree of (musical) consonance of a chord is related to the distribution of potential root tones of a chord. Hereby, the root tone can be seen as the tone with the maximum static attraction given the chord as musical context. In cases with a single, prominent root tone, the chord sound more consonant than in cases where several root tones are in competition. Formally, we can explain the degree of consonance of a chord as the static attraction value of the (root) tone with maximum attraction after normalizing the attraction profile (i.e., the attraction values of the 12 tones sum up to 1).

The mirror symmetry (against the triton) of the spatial deformation model leads to important problems when it comes to account for the difference between major and minor modes. Important differences between major and minor modes were discussed already 90 years ago (Heinlein, 1928). Recently, Johnson-Laird, Kang, and Leong (2012) have investigated chords including major triads (CEG), minor triads (CE♭G), diminished triads (CE♭G♭), and augmented triads (CEG#). The following table shows the empirical ratings of the chord's consonance. Clearly, the major chords exhibit the highest degree of consonance followed by the minor chords. Further, the diminished chords are ranked lower and, at the bottom, we (surprisingly) find the augmented chords. It is not difficult to see that the hierarchic model and the symmetric deformation model predict the same degrees of consonance for major and minor chords.

Triad Class	*Empirical Consonance Rating*	*Hierarch. Model*	*Asymmetric Deformation Model*	*Asymmetric Phase Model*
major	5.33	.49	.54	.50
minor	4.59	.49	.52	.49
diminished	3.11	.34	.36	.45
augmented	1.74	.33	.34	.35

Table 5: Empirical rankings and model predictions for common triads. The predictions of the models concern the strength of strongest static attraction using normalized attraction profiles.

In order to model these data, we have considered modifications of the deformation model and the phase model. Both model variants can explain the correct ranking of the four chord types. This is shown in the two columns on the right hand side of Table 5. Beim Graben & Blutner (2018) have considered an asymmetric deformation model with a deformation polynomial of fourth order, where the linear term and the term of third order break the mirror-symmetry against the tritone F#, see formula (40)a. For the asymmetric model based on phase shifts, we move the center $\check{c}$ of the symmetry from π a bit to the left in order to break the symmetry as shown in (40)b.

(40) a. $\gamma(x) = a + b(x-\pi)^4 \;\rightarrow\; \gamma(x) = a + b(x-\pi) + c(x-\pi)^2 + \frac{b}{\pi^2}(x-\pi)^3 - \frac{a+c\pi^2}{\pi^4}(x-\pi)^4$

b. $\delta(x) = \check{a} + \check{b}(x-\pi)^2 \;\rightarrow\; \delta(x) = \check{a} + \check{b}(x-\pi+\check{c})^2$.

In both cases, the three parameters have been fitted after inserting the kernel functions corresponding to $\gamma(x)$ and $\delta(x)$, respectively, into the convolution equation (7) for the experimental C major data. The resulting kernel functions are presented in Fig. 11 (left hand side for asymmetric deformation, right hand side for asymmetric phase shifts).

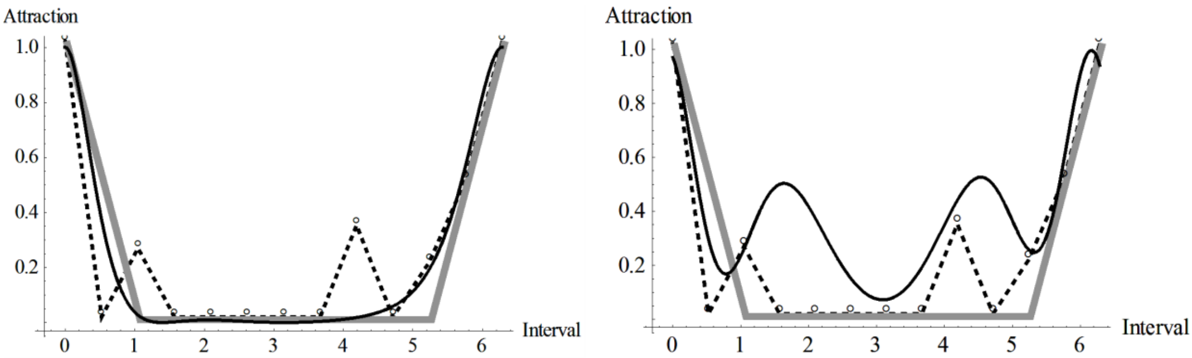

Figure 11: Three kernel functions. Grey bold line: Hierarchical model; Dashed line: Purncutt's asymmetric (template-based) model; Solid line: asymmetric quantum kernel (left hand side: deformation model; right hand side: phase model)

Fig. 11 shows that both models approximate the asymmetric flanks of Parncutt's (1989) model (Note that the two small peaks localized near 1.2 and near 4.3 are not very important for the predictions of the model). The predictions of the fit with Krumhansl's & Kessler's (1982) C-major data for static attraction are important. For the deformation model, the introduction of asymmetry improves the correlation value from r = .97 to r =.982 . For the phase model, the introduction of asymmetry improves the correlation value from r = .855 to r = .903. In both cases, the ranking of the consonance values agrees with the empirical ranking. However, the strength of the differences between major and minor modes and between diminished and augmented modes is considerably less than the data suggest.

Summarizing, we have considered symmetry breaking in cognitive musicology, breaking the mirror symmetry of the tonal attraction kernel. In this way, we have overcome some weaknesses of the classical attraction model based on tonal hierarchies. This model cannot even account for the differences between major and minor modes. Consequently, we have constructed a model that accounts not only for static attraction profiles but also for graded consonance/dissonance. The ability for unification – grasping different phenomena in a systematic way – is one of the trademarks of quantum theory. It is correspondingly visible also in the domain of quantum cognition.

5 Discussion and Conclusions

In this article, I have contrasted the metaphoric and the realist conceptions of musical forces. Both approaches provide interesting perspectives to study static and dynamic tonal attraction profiles, including their structure, use, and acquisition. The metaphoric conception is taken from mainstream cognitive psychology initiated by work of Lakoff and Johnson (Lakoff, 1987; Lakoff & Johnson, 1980, 1999). The realist conception of musical forces is a new development within the evolving field of quantum cognition and sees musical forces as gauge forces. Here is a short presentation of both approaches.

Let us start with the metaphoric conception. Notably, a *conceptual metaphor* refers to the understanding of one conceptual field, in terms of another. Famous examples illustrating the idea are "life is a journey" or "time is money". According to this concept, musical forces are constructs *in analogy* to our understanding of physical forces in folk physics. Various authors have proposed different forces, which are assumed to be important for musical perception. Linear regression analysis has been applied in order to find the total effect of musical forces. Unfortunately, the realized fits with empirical data are not really convincing. A sound grounding of forces seems not to be possible in this way.

In contrast, the realist conception of musical forces constructs these forces as gauge forces, which can be derived from fundamental symmetries of the underlying theory and a gauge field. In the present case, we model tones by vectors of a 2-dimensional spinor Hilbert space. Hence, the basic symmetry group is the group of special unitary transformations (SU(2) and their subgroups). If you want, the proposed realist conception is a fresh realization of Kepler's 400 years old vision of *harmonices mundi* (Kepler, 1619) – the vision unifying physics and musicology. Recognizing the failing of Kepler's ideas – mainly for reasons the concern certain astronomical facts not known yet 400 years ago – the present approach is based on a totally different scenery centered around the insights of modern quantum cognition (e.g., Busemeyer & Bruza, 2012).

We have identified the symmetry group SU(2) as the fundamental group that directs the gauge theoretic approach. For reasons of simplicity, we did not consider the full group but two subgroups only, the rotation group SO(2) and the phase shifting group U(1). This procedure of not exploiting the full complexity of the physical world but allowing abstractions and idealizations is quite common in physical sciences (Ilgenfritz et al., 2015). The analysis of the SO(2) subgroup straightforwardly leads to the deformation model, and the analysis of the unitary subgroup conforms to what I call the phase model. In a nutshell, the deformation model and the phase model, both give a fairly adequate description of the available data discussed in this paper even when I have to add that the deformation model seems a bite more adequate. The calculation of potential densities shows large differences for the two models. Since these data are difficult to interpret empirical, a clear position which model should be preferred is missing at the moment. I regard it as very good advice to explore this aspect in future work.

Concerning the results of the gauge theoretic modelling attitude, I will stress three aspects which can be seen as main characteristics of the present research. First, the idea of discrete convolution – an operation that describes the modification of a kernel function by a distribution of several contextual elements (for instance, the tones of a single chord). Second, the present approach establishes a close connection between static and dynamic attraction. This is visible in both gauge models. The basic idea of connecting static and dynamic attraction goes back to Ball (2010). It was concretized in the present article. Third, the notion of energy densities and musical forces was a particular outcome of the gauge theoretic approach. These concepts are directly connected with the idea of dynamic stability as established in the field of dynamical systems.

Finally yet importantly, I should mention several unresolved issues, which should be essential points for further research. A first topic is the innateness issue. Already Leonard Bernstein has vehemently disputed the issue expressing and stressed the point that musical apperception is not possible without an innate cognitive background. At the end of his Norton lectures, he formulates his deep believe in the tonal system in the following magical phrases:

> I believe that from that Earth emerges a musical poetry which is by the nature of its sources tonal.
> I believe that these sources cause to exist a phonology of music, which evolves from the universal known as the harmonic series.
> And that there is an equally universal syntax, which can be codified and structured in terms of symmetry and repetition. (Bernstein, 1976)

In the present context, this innate background is mainly constituted by the tonal kernel function and operations such as discrete convolution that are not acquired by learning.

A second issue concerns the full dynamic attraction potential, which does not describe the resolution of chords only but the full development of melodies. Besides the metaphoric approaches I have mention in Sect. 2 there are approaches based on Bayesian networks (Temperley, 2008). The present approach suggest to exploit the dynamic evolution described by the Schrödinger equation, which can be seen as a generalization of the Kolmogorov forward equation (Busemeyer & Bruza, 2012).

A third big question, which was not fully treated in this article, concerns the nature of the distinction between consonance and dissonance. For building the distinction, learned parameters seem to play a much more important role than usually assumed (McDermott, Schultz, Undurraga, & Godoy, 2016). Further, a developed theory of this distinction is important for defining the 'meaning of music' and its emotional content (Blutner, forthcoming).

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