Toward a Gauge Theory of Musical Forces

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Abstract. How well does a given pitch fit into a tonal scale or key, being either a major or minor key? This question addresses the well-known phenomenon of tonal attraction in music psychology. Metaphorically, tonal attraction is often described in terms of attracting and repelling forces that are exerted upon a probe tone of a scale. In modern physics, forces are related to gauge fields expressing fundamental symmetries of a theory. In this study we address the intriguing relationship between musical symmetries and gauge forces in the framework of quantum cognition.

1 Introduction

The application of physical metaphors is quite common in theories of tonal music. The basic assumption seems to be that our experience of musical motion is in terms of our experience of physical motion and their underlying forces. For example, Schönberg speaks of different forces when he explains the direction of musical forces in cadences where the tonic attracts the dominant [23, p. 58]. In a similar vein, Larson [14] proposed three musical forces generating melodic completions, which he calls *gravity*, *inertia*, and *magnetism*, respectively. These forces should be regarded as conceptual metaphors in the sense of Lakoff and Johnson [12]. They structure musical cognition in analogy with falling, inert and attracting physical bodies. Physical forces are represented in our naive (common sense) physics or folk physics.

In contrast to Larson, Mazzola [18] suggested a quite different analogy between music theory and modern (non-folk) physics. Modern foundational physics describes forces as being caused by the "exchange" of particular particles. Forces are basically connected with certain symmetries of the physical micro-world. Mazzola was probably the first who saw the analogy between physics and music in connection with the existence of musical symmetries, especially for the domain of modulation. Although Mazzola did not directly apply quantum theory for his theoretical models, he made use of a simplified framework for handling the underlying symmetries.

Mazzola's insights are of highest importance for the present paper, because "exchange particles" in the standard model of elementary particle theory emerge from the quantization of gauge fields mediating symmetry transformations between localized quantum states. Therefore, we investigate the central problem of tonal attraction in terms of quantum symmetries and gauge fields. The term "tonal attraction" refers to the idea that melodic or voice-leading pitches tend toward other pitches in greater or lesser degrees. The present conception sees a close relationship between the phenomenon of tonal attraction and the existence of tonal forces. After a short discussion of the music-psychological phenomenon of tonal attraction in the next section, Sec. 3 provides a quantum-cognitive model based on a qubit representation of tones along the lines developed in [3]. In Sec. 4 we outline a gauge theory of musical forces, presenting first the force-free case as a default model which essentially reproduces the findings of the qubit model. Second, it is demonstrated how the introduction of local phase factors can improve the descriptive power of the model. Gauge forces can be regarded as corrections terms that apply to the force-free (default) case. Section 5, finally derives some general conclusions and gives an outlook on future works, e.g. the possible relationship of gauge theory and brain wave models [21] similar to existing proposals by de Barros and Suppes [1], Large [13], and most recently Friston and coworkers [25].

2 The Phenomenon of Tonal Attraction

In the last twenty years, there has been an enormous progress in the development of cognitive theories of tonal music. A central issue has been the question of tonal attraction. How well does a given pitch fit into a tonal scale or tonal key, let it be a major or minor key? In a celebrated study, Krumhansl and Kessler [11] asked listeners to rate how well each note of the chromatic octave fitted with a preceding context, which consisted of short musical sequences in major or minor keys. This finding plays an essential role in Lerdahl's and Jackendoff's generative theory of tonal music [16] and is one of the main pillars of the structural approach in music theory.

For illustration, Fig. 1(a) depicts the C major scale arranged around the circle of fifths comprising 12 semitones within one octave. The tonic, indicated with "0", defines the origin of the chroma circle [9]. Open bullets are members of the C major (diatonic) scale, while black bullets do not belong to the scale. One can see from Fig. 1(a) that the whole chromatic scale is divided into two connected subparts: the diatonic part (open bullets) and the remaining (nondiatonic) part. The empirical results of Krumhansl and Kessler [11] are replicated in Fig. 1(b) for the C major context. The probe tones are represented as real numbers $x = j\pi/6$ ($j = 0, \dots 12$, with C(0) $\cong$ C'(12) one octave higher) at the x-axis corresponding to the radian angles at the chroma circle Fig. 1(a). The subjective ratings $y(x)$ are plotted at the y-axis. The results of this experiment clearly show a kind of hierarchy: the tonic pitch $j = 0$ which is mostly attracting received the highest rating, followed by the pitches completing the tonic triad (third $j = 1$ and fifth $j = 4$), followed by the remaining scale degrees, and finally the chromatic, nonscale tones.

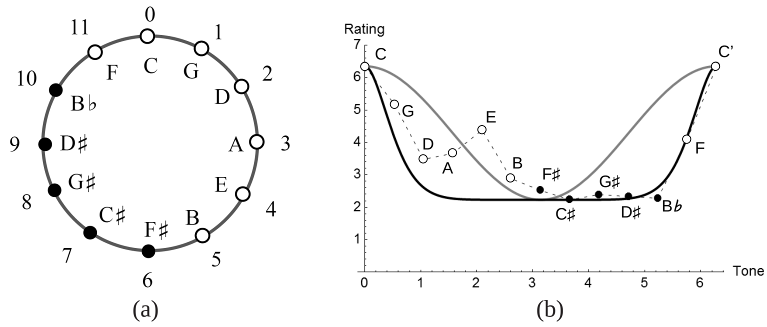

Fig. 1. Tonal attraction at the chroma circle. (a) The circle of fifths for C major scale as indicated by open bullets. (b) Rating data $y(x)$ of Krumhansl and Kessler [11] (dashed-bullets) and scaled quantum models of tonal attraction. Gray bold-solid: unmarked quantum model from Sec. 4.1 [Eq. (9)]. This model makes the same predictions as the qubit quantum model described in Sec. 3. Black bold-solid: marked quantum model from Sec. 4.2 [Eq. (11)]. Obviously, the nondiatonic pitches (6 – 10 on the circle of fifth) are the pitches with the lowest attraction values as described by the traditional, hierarchic model.

3 Qubit Quantum Model of Tonal Attraction

One important model for the Krumhansl and Kessler [11] data was given by Lerdahl [15] and recently rephrased by Blutner [3] in terms of optimality theory [22]. In this framework, cognitive representations are described by several constraints that could either be satisfied of violated. The constraint violation profile of a construction accounts for its *markedness*. Unmarked constructions are generally easier to process in psychological experiments as is reflected by lower processing times and higher accuracies. On the other hand, marked constructions increase processing demands in terms of "mental energy" or "cognitive forces". Therefore it sounds reasonable to look for a similar relationship between tonal markedness in the sense of [3, 15] and musical forces.

One of the fundamental ideas of quantum cognition is to apply the mathematics of the physical formalism to the domain of cognition. For example, we can use a series of qubit states to represent the 12 pitch classes used in tonal music. In addition, we can use the probability that one of these qubit state collapses into another one as a measure for the tonal attraction between the corresponding tones (see [3]).

For getting an explicit model of tonal states as states of a Hilbert space, the concept of symmetry is essential. Mathematically, symmetry is simply a set of transformations applied to given states such that the transformations preserve the properties of the states. In music, the most basic symmetry principle is the *principle of translation invariance*. It says that the musical quality of an episode is essentially unchanged if it is transposed into a different key. That means, the operations of the cyclic group $\mathbb{Z}_{12}$ are applied to the chroma circle from Fig. 1(a) [18]. Therefore, we can say that $\mathbb{Z}_{12}$ is the symmetry group of (Western) music.

More concretely, in the present case of tonal music, the underlying symmetry group could be represented by certain rotations of vectors in a two-dimensional vector space. For instance we can rotate the vector $\varphi_{\rightarrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in n steps to the original vector. In linear algebra, the elementary rotation steps can be described by the following rotation matrix γ:

$$\gamma = \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \tag{1}$$

Performing a repeated application of the rotation matrix to our vector $\varphi_{\rightarrow}$ above, we can generate the 12 tones of the circle of fifth in the following way:

$$\psi_j = \gamma^j \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\frac{\pi j}{12} \\ \cos\frac{\pi j}{12} \end{pmatrix} \tag{2}$$

In the case of pure states, quantum theory defines structural probabilities. This means the probability that a state ψ collapses into another state depends exclusively on the geometric, structural properties of the considered states. How well does a given tone fit with the tonic pitch? What is the probability that it collapses into the (tonic) comparison state? The probability of a collapse of the state ψ_j into a state ψ_l can be calculated straightforwardly:

$$p_{\psi_l}(\psi_j) = \cos^2\frac{\pi(j-l)}{12} = \frac{1}{2}\left[1 + \cos\frac{\pi(j-l)}{6}\right] \qquad \text{where } 0 \leq j, l < 12\,. \tag{3}$$

For a fixed element ψ_l the probabilities of the 12 tones indexed by $j (0 \leq j < 12)$ sum up to one. Hence, formula (3) offers a probabilistic attraction profile relative to a given context tone ψ_l to which we refer to as a *kernel function*. If the context is not given as a single tone, but rather as a tonal region, a chord, or a series of chords, then we would consider the mixture of all the states conforming to all the involved single tonal elements. For simplicity, we could take all tones contributing to this mixture as being equivalent and give them the common weight $1/N$ (assuming N tonal elements are taken into account), thereby computing a density operator over different kernel functions [3]. This assumption is rather similar to Woolhouse's treatment of the problem of context effects in tonal attraction [26].
Figure 1(b) shows the attraction profile for the C major key as the kernel $p_{\psi_0}(\psi_j)$, obtained from the quantum model, and scales it to the Krumhansl and Kessler data [11] plotted in gray bold. Note that the quantum model is parameter-free. The correlation coefficient between the predicted profile and the Krumhansl-Kessler profile is $r = 0.7$ in the case of C major. That means that about 50% of the variance is already explained by the default quantum kernel.
In order to permit the comparison with the symmetric model of Woolhouse [26], we fitted a kernel mixture, assuming symmetric phase parameters in the quantum model (i.e., the phases of the first seven tones of the circle of fifth are mirrored at the tritone point). The phase parameters were fitted as follows (starting from the tonic in the circle of fifth): $(0, \pi/2, \pi, 0, 0.9, 0, 0.99, 0, 0.9, 0, \pi, \pi/2, 0)$. In the present case of a symmetric kernel function, the correlation coefficient between the model fit and the Krumhansl-Kessler profile is $r = 0.82$ in the case of major keys. Moreover, an asymmetric distribution of phase angles improves the goodness of fit to $r = 0.95$ for major keys [3].

4 Gauge Theory of Tonal Attraction

In the following, we present an alternative treatment of tonal structures allowing the introduction of musical forces that is inspired by quantum gauge theory. In the last paragraph we have seen how the introduction of locally different phase factors could substantially improve the goodness of fit of the quantum model of tonal attraction. In modern physics, such phase functions lead naturally to the emergence of forces as frustrated connections of an underlying spatial structure.
In contrast to the qubit approach explained above, where a tone was represented by a state in the Hilbert space $\mathscr{H} = \mathbb{C}^2$ subjected to the cyclic group $\mathbb{Z}_{12}$ as symmetry, we strive here for a representation in terms of Schrödinger wave functions. A wave function is a state in a function Hilbert space $\mathscr{H} = L^2(\Omega)$ of complex-valued (square-integrable) functions $\psi : \Omega \to \mathbb{C}$ over a configuration space Ω. I.e., for a fixed "site" $x \in \Omega$, the value $\psi(x)$ belongs to a "local" Hilbert space $\mathscr{H}_x = \mathbb{C}$ attached to x. These local Hilbert spaces altogether form a "fiber bundle" over the configuration space Ω, which is the appropriate framework of gauge theory as required for the proper treatment of musical forces.
Our starting point is the chroma circle Fig. 1(a) representing tones as equivalence classes of pitches over one octave. This is essentially the continuum of the unit circle $S^1 = \mathbb{R} \pmod{2\pi}$ which contains the semitone cyclic group $\mathbb{Z}_{12}$ as a subgroup. A tone is then given through its radian angle $x = j\pi/6$ ($j = 0, \ldots 12$) as a spatial site of the unit circle. Therefore, the "tonal configuration space" of our quantum model will be taken as the chroma circle $\Omega = S^1$. A quantum state is then given as a wave function $\psi(x,t)$ that is dependent on tonal site $x \in S^1$ and time t solving the one-dimensional Schrödinger equation [24]

$$H\psi = \mathrm{i}\frac{\partial \psi}{\partial t} \tag{4}$$

with Hamilton operator H.[3] Finally, the complex value of a wave function $\psi(x,t)$ for fixed x, t will be regarded as a state in a local Hilbert space $\mathscr{H}_x = \mathbb{C}$ allowing for gauge transformations.

4.1 Unmarked Behavior

In a first approximation for the unmarked behavior, we study the "movement" of a free particle with Hamiltonian

$$H = T = p^2$$

around the chroma circle. Here T denotes kinetic energy with $p = -\mathrm{i}\partial/\partial x$ the momentum operator. Inserting the latter expressions into Eq. (4) yields

$$-\frac{\partial^2 \psi}{\partial x^2} = \mathrm{i}\frac{\partial \psi}{\partial t} \tag{5}$$

which is solved by plane waves

$$\psi_k(x,t) = A_k \mathrm{e}^{\mathrm{i}(kx - \omega t)} \tag{6}$$

[3] Note that we chose a natural unit system with particle's mass $m = 1/2$ and Planck's quantum of angular momentum $\hbar \equiv 1$ as necessary for quantum cognition applications.

and their linear combinations, where A_k denote complex amplitudes. The wave number k and circular frequency ω depend on each other through the dispersion relation

$$\omega = k^2 \,. \tag{7}$$

The final solution of the Schrödinger equation must obey the given initial and boundary conditions. As initial condition we may set $\psi(x_l, 0) = A$ for encoding the tonic of the context as a phase shift x_l with A as maximal attraction amplitude that is subjected to the normalization constraint

$$\int_\Omega |\psi(x,t)|^2 \, \mathrm{d}x = 1 \,.$$

Additionally we need Möbius-type periodic boundary conditions on the unit circle $\psi(x + 4\pi, t) = \psi(x,t)$ thus reflecting the double covering from Eq. (2). Therefore, the chroma circle exhibits the topology of a Möbius tape. Interestingly, Mazzola [18] has visionarily foreseen the putative relevance of these structures for mathematical music theory as well. Moreover, Möbius-type connectivities have been suggested as possible organizational principles of cortical structure and brain wave dynamics by Wright and coworkers [27, 28]. The former yields the normalization $A = 1/(2\sqrt{\pi})$, while the latter gives a quantization constraint $\mathrm{e}^{4\pi \mathrm{i} k} = 1$, and hence $k \in \mathbb{Z}/2$. Choosing the two fundamental wave numbers $k = \pm 1/2$ yields $\omega = 1/4$ and $A_k = A\mathrm{e}^{-\mathrm{i} k x_l}$. Finally, the superposition of fundamental solutions entails

$$\psi(x,t) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\mathrm{i}\frac{t}{4}} \cos \frac{x - x_l}{2} \tag{8}$$

which is a standing wave along the unit circle with probability density

$$p(x) = |\psi(x,t)|^2 = \frac{1}{\pi} \cos^2 \frac{x - x_l}{2} \,. \tag{9}$$

Inserting the semitones $x_j = j\pi/6$ around the circle of fifths for x, confirms the previous result obtained from the qubit quantum model [3] (Sec. 3).

$$p_j(x_l) = |\psi(x_j,t)|^2 = \frac{1}{\pi} \cos^2 \frac{\pi(j - l)}{12} \,. \tag{10}$$

This default distribution kernel characterizes *unmarked* music cognition and is plotted in gray bold after scaling in Fig. 1(b). The correlation with the Krumhansl-Kessler data [11] is $r = 0.7$ as reported above.

4.2 Marked Behavior

In order to understand marked behavior as well, we have to develop a theory of musical forces that complements the metaphoric notions of Larson [14] and Mazzola [18]. To that aim, we first realize that the distribution (10) simply reflects the similarity relations between tones along the chroma circle where C, G, and F are close neighbors and hence similar with respect to their attraction profiles, whereas C and the tritone F♯ are maximally distant and thus unrelated [Fig. 1(a)]. A suitable deformation of the distances along the chroma circle could lead to an improved description of the empirical data presented in Fig. 1(b). Therefore, we make the ansatz

$$\psi(x) = A\cos(\gamma(x)) \tag{11}$$

for the stationary wave function where $\gamma(x)$ is a spatial deformation function and A a normalization constant. For the sake of simplicity, we focus on the C major scale with $x_l = 0$ here. Differentiating (11) twice and eliminating trigonometric terms, we obtain the differential equation

$$-\psi''(x) + \frac{\gamma''(x)}{\gamma'(x)}\psi'(x) - \gamma'(x)^2\psi(x) = 0 \tag{12}$$

which we compare with the stationary Schrödinger equation $H\psi(x) = E\psi(x)$ for the energy eigenvalue E. With

$$H = T + M + U$$

this comparison yields the following operators: The first term T is, as usual, the operator of kinetic energy

$$T = -\frac{\partial^2}{\partial x^2}\,.$$

The second term could be interpreted in the context of electromagnetism where the velocity-dependent contribution to the Hamilton operator is regarded as magnetic interaction energy

$$M = \frac{\gamma''(x)}{\gamma'(x)}\frac{\partial}{\partial x}$$

Finally, the last term, which is simply a scalar multiplication operator, receives its usual interpretation as potential energy

$$U = E - \gamma'(x)^2$$

which might be seen either as electrostatic or gravitational potential. Note that the constant

$$E = \gamma'(0)^2 \tag{13}$$

can be interpreted as the total energy of the tonal dynamics.
The marked Schrödinger equation obeys conservation of energy, as unveiled by multiplication with the adjoint solution ψ^* from the left. Introducing energy densities

$$t(x) = -\psi(x)^*\psi''(x) \tag{14}$$

$$m(x) = \psi(x)^*\frac{\gamma''(x)}{\gamma'(x)}\psi'(x) \tag{15}$$

$$u(x) = \psi(x)^*(E - \gamma'(x)^2)\psi(x) \tag{16}$$

yields

$$t(x) + m(x) + u(x) = E\psi(x)^*\psi(x) = E\,p(x)$$

with $p(x) = |\psi(x)|^2$ the resulting probability distribution. Interestingly, this distribution describes the original Krumhansl-Kessler data [11] which therefore receive a straightforward interpretation as *total energy density* of tonal attraction.

From the general deformation ansatz Eq. (11) for the marked case we retain the unmarked wave function by the choice

$$\gamma_u(x) = \frac{x}{2}$$

rendering the force-free dynamics with $U(x) = E - \gamma_u'(x)^2 = E - 1/4$, i.e. $U = 0$ and $E = \omega = 1/4$. For the marked attraction profile we assume a symmetric polynomial of fourth order

$$\gamma_m(x) = a_0 + a_4 (x - \pi)^4 \,,$$

with boundary conditions $\gamma(0) = 0$, i.e. the tonic should not be deformed, and $\gamma(\pi) = \pi/2$, i.e. the tritone receives maximal deformation. This leads to the parameter-free model

$$\gamma_m(x) = \frac{\pi}{2} - \frac{(x-\pi)^4}{2\pi^3} \,. \tag{17}$$

Interestingly, the terms higher than linear order can be interpreted as spatially dependent phase shifts of the unmarked wave function which depends on the linear term only. From (17) we obtain the total energy (13) as $E = 4$ which is sixteen times larger than the energy required to the unmarked dynamics.
Inserting the deformation (17) into the wave function (11), yields the marked attraction kernel for the tonic context, plotted as the bold black curve in Fig. 1(b). The correlation with the Krumhansl-Kessler data [11] is $r = 0.89$, i.e. our fit accounts now for 79% of the data's variance. Computing the mixture over the C major tonic triad context, improves the fit to $r = 0.97$, covering 95% of the data.
Finally, we compute the three energy densities (14 – 16) and also the density of potential energy alone

$$d(x) = m(x) + u(x) \,. \tag{18}$$

The results are presented in Fig. 2.

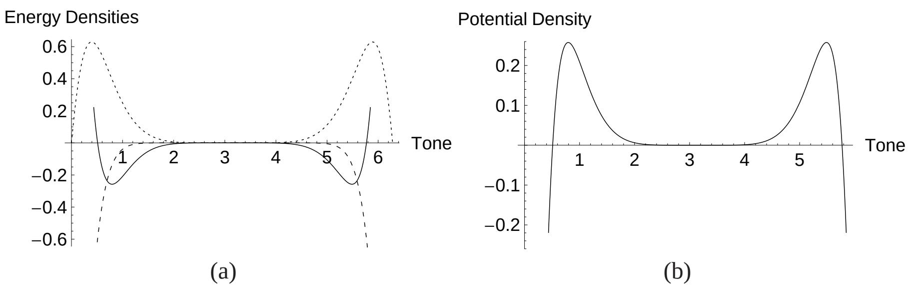

Fig. 2. Emergent energies of tonal attraction. (a) Energy densities: Solid: "inertia" $i(x)$, dashed: "gravity" $u(x)$, dotted: "magnetism" $m(x)$. (b) Density of potential energies $d(x)$ [Eq. (18)].

Figure 2(a) shows the three densities "inertia" (solid), "gravity" (dashed), and "magnetism" (dotted). Both, "inertia" and "gravity" clearly indicate that the tonic at $x = 0 \pmod{2\pi}$ acts as a center of gravity, where the gravitation potential (16) approaches minus infinity while the kinetic energy (14) tends toward plus infinity. Therefore, the gravitational force which is the negative gradient of the potential is negative in the tonic's vicinity, i.e. the tonic is attracting, leading to high acceleration (14). The tonic is also attracting with respect to the "magnetic" force (15) which also has positive slope for small x-values. However, for tones in the interval $0.4 < x < 5.8$, corresponding to G – F, "magnetism" prevents tones from being attracted by the tonic. This makes the tritone F♯ a "magnetic trap" in this region.

Even more instructive is Fig. 2(b) depicting the summed potential energy density. Here again, the tonic appears as a center of force. Extrema of the potential $d(x)$ are equilibrium points which are either unstable for local maxima or stable for local minima. On the one hand, there are two unstable equilibria around $x = 0.8$ (D) and $x = 5.4$ (B♭). On the other hand, the only equilibrium at $x = \pi$ is stable, which is precisely the tritone. Because the total energy density is low in this region, tones are trapped by the tritone.

4.3 Gauge Invariance

Finally, we have to prove the local gauge invariance of our music quantum model. To that aim, we first realize that the probabilities $p(x)$ do not change under a shift of the wave functions's phases. Let ψ be an arbitrary wave function solving the Schrödinger equation (4) and $\varphi \in \mathbb{R}$ a real phase value. Then the operation $\psi \mapsto \tilde{\psi} = \mathrm{e}^{\mathrm{i}\varphi}\psi$ yields another solution of the Schrödinger equation simply obtained by multiplying Eq. (4) with $\mathrm{e}^{\mathrm{i}\varphi}$. However, this *global gauge transformation* does not affect the observable probabilities $\tilde{p} = |\tilde{\psi}|^2 = p$.

Yet, things get much more involved when the phase shift becomes a function of space,[4] $\varphi(x)$, describing a *local gauge transformation*. Writing

$$\tilde{\psi}(x) = \mathrm{e}^{\mathrm{i}\varphi(x)}\psi(x) \tag{19}$$

we have to take the spatial derivatives in (12)

$$\frac{\partial \tilde{\psi}}{\partial x} = \mathrm{i}\frac{\partial \varphi}{\partial x}\mathrm{e}^{\mathrm{i}\varphi}\psi + \mathrm{e}^{\mathrm{i}\varphi}\frac{\partial \psi}{\partial x} = \mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial \varphi}{\partial x}\right)\psi\,.$$

Repetition of the derivation yields the Laplacean

$$\frac{\partial^2 \tilde{\psi}}{\partial x^2} = \frac{\partial}{\partial x}\left[\mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial \varphi}{\partial x}\right)\psi\right] = \mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial \varphi}{\partial x}\right)^2\psi\,.$$

For the operator appearing in round brackets we introduce the notation

$$\mathrm{D}_x = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial \varphi}{\partial x} \tag{20}$$

which is called *covariant derivative*, thereby alluding to the curved space of general relativity which was the historically first formulated *local gauge theory*. The gradient of the phase function $\varphi(x)$ is called the *gauge field* in this connection.

The Schrödinger equation (12) is called *locally gauge invariant*, if the transformed wave function obeys a structurally equivalent equation with transformed coefficients

$$-\tilde{\psi}''(x) + \frac{\tilde{\gamma}''(x)}{\tilde{\gamma}'(x)}\tilde{\psi}'(x) - \tilde{\gamma}'(x)^2\tilde{\psi}(x) = 0\,. \tag{21}$$

[4]For the sake of simplicity, we neglect time-dependence of the gauge field in our exposition.

Using covariant derivatives instead of the conventional ones (which emerge as limiting cases for $\varphi = \text{constant}$) yields

$$-\mathrm{D}_x^2 \psi(x) + \frac{\tilde{\gamma}''(x)}{\tilde{\gamma}'(x)} \mathrm{D}_x \psi(x) - \tilde{\gamma}'(x)^2 \psi(x) = 0 \,,$$

which gives after some rearrangements

$$-\psi''(x) + \left(\frac{\tilde{\gamma}''(x)}{\tilde{\gamma}'(x)} - 2\mathrm{i}\varphi'(x) \right) \psi'(x) - \left[\tilde{\gamma}'(x)^2 - \varphi'(x)^2 + \mathrm{i} \left(\varphi''(x) - \frac{\tilde{\gamma}''(x)}{\tilde{\gamma}'(x)} \varphi'(x) \right) \right] \psi(x) = 0 \,.$$

This expression is invariant under the constraints

$$\frac{\tilde{\gamma}''(x)}{\tilde{\gamma}'(x)} - 2\mathrm{i}\varphi'(x) = \frac{\gamma''(x)}{\gamma'(x)} \tag{22}$$

$$\tilde{\gamma}'(x)^2 - \varphi'(x)^2 = \gamma'(x)^2 \tag{23}$$

$$\varphi''(x) = \frac{\tilde{\gamma}''(x)}{\tilde{\gamma}'(x)} \varphi'(x) \,, \tag{24}$$

which restrict the freedom of choice for the local phase function $\varphi(x)$. Thus, our musical gauge theory has a broken symmetry that is not the full U(1) symmetry of quantum electrodynamics.

5 Discussion and Outlook

In this study we have discussed the phenomenon of tonal attraction in a quantum cognition framework. After reviewing a previous approach based on a qubit representation of the essential musical symmetry group [3], we formulated an alternative description in terms of wave functions. Solving the Schrödinger equation of a "free particle" over the circle of fifths as musical configuration space, we were able to reproduce the results of the unmarked qubit quantum model for the experimental findings of Krumhansl and Kessler [11]. In a second step we addressed the important issue of gauge symmetry of the Schrödinger equation and derived three expressions for musical forces which might be related to similar concepts discussed in the literature [14, 23]. The introduction of gauge forces led to a spatial deformation of the circle of fifths that we approximated by a polynomial of fourth order, for which we could explicitly derive the musical forces of tonal attraction of the marked quantum model, in good agreement with the Krumhansl and Kessler data.

Sofar, our approach accounts for the effect of "static forces" which determine the center(s) of a series of tones or chords by means of stationary wave functions. Yet, there are also "dynamic forces" affecting melodic or harmonic progression and predictability, investigated, e.g. in [10]. The most interesting dynamical aspect of music theory is, notably, modulation, the dynamic transition from one scale or key into another one. Inspired by Schönberg's modulation theory [23], Mazzola [18] developed a sophisticated mathematical account based on musical symmetries and cadences. Its most important ingredient is, what he calls the "modulation quantum", a collection of chords mediating the dynamic transition from one key into another. It will be a challenging endeavor to further develop our gauge theory of musical forces into these fascinating directions.

In the recent literature of explaining tonal attraction, the spectral pitch class model [20] plays an essential role. In this model, the pitch perception of any musical sound is described by using spectral pitch class vectors. There are close similarities between this Helmholtzian model [8] and the present quantum approach which should be pursued in a later publication. At this point we only note that Schrödinger's idea of "quantization as eigenvalue problem" [24] was crucially influenced by Helmholtz' idea of oscillating strings.
Next, let us speculate about the putative relevance of our approach in the neurosciences. Partial differential equations are well-known in the discipline of neural field [5] and dynamic neural field theory [7, 17] within computational neuroscience where they appear as brain wave equations [5, 21]. In the latter, fields are regarded as functions over *abstract feature spaces* and we might consider the chroma circle in our approach as such a feature space. These neural fields are clearly real-valued functions in contrast to the generically complex wave functions solving the Schrödinger equation. However, according to Bohm [4], the Schrödinger equation for one complex field is equivalent to two coupled real fields describing the motion of a classical particle in a "quantum mechanical potential" and its respective field dynamics. In quantum theory this leads to disputably nonlocal representations. Yet in neural field theory, nonlocal interactions are ubiquitous due to long-range synaptic connectivity. Thus, our gauge theory of musical forces may find its neurophysiological counterparts in the organization of cortical areas [25, 28].
Finally, let us remark on the relationship between the process of musical perception and the musical composition process. A very naive understanding of the composition process is that it is nothing else than looking for the most probable continuation of a starting sequence of tones. Of course, this is simply to realize with the help of neural networks (e.g. [2]). A composer normally aims to generate emotions in the mind of the listener. Emotions are deeply connected with subjective expectancy [19]. However, it is crucially surprise that generates great musical effects. Hence, the process of composition cannot be described as a mechanism for finding the most probable continuation. If one insists to view the process of composing as an optimization algorithm, then one has to considering higher rules of optimization. These rules are directed to resolving conflicting aims in following particular emotional goals, optimally separating different voices and, at the same time, pursuing certain restrictions of a particular style.

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