Fuzzy logic

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1 Fuzzy Sets and Logic

Uncertainty and vagueness should be sharply distinguished. It is not certain what will be the outcome of the next throw of the die I hold in my hand. There is nothing vague about the outcome, however–it is one of the numbers $\{1, \ldots, 6\}$. On the other hand, in real-life applications of AI such as medicine, outcomes are not always sharply defined. For instance, in the expertsystem MUNIN, which assists doctors in diagnosing muscular diseases, there occurs a variable called 'muscular loss' whose values are {no, moderate, severe, total, other}. Clearly 'moderate' or 'severe' are not sharply delineated concepts; they are instances of vagueness. Often a single piece of knowledge about a domain abounds with expressions both for uncertainty and for vagueness. MUNIN provides one example; here is another, from a textbook on cancer. Vague expressions are given in bold type, expressions for (un)certainty in italics.

> **Chronic** cystic disease is *often* confused with carcinoma of the breast. It *usually* occurs in **parous** women with **small** breasts. It is present *most commonly* in the **upper outer quadrant** but *may* occur in other parts and eventually involve the **entire** breast. It is *often* **painful**, particularly in the premenstrual period, and accompanying **menstrual disturbances** are *common*. Nipple discharge, *usually* servous, occurs in *approximately 15% of the cases*, but there are no **changes** in the nipple itself. The lesion is **diffuse without sharp demarcation** and without fixation to the overlying skin.. Multiple cysts are **firm, round and fluctuant** and *may* transilluminate *if* they contain a **clear** fluid. A **large** cyst in an area of chronic cystic disease **feels like a tumor**, but is *usually* **smoother and well-delimited**. The axillary lymph nodes are *usually* not **enlarged**. **Chronic** cystic disease *infrequently* shows **large bluish** cysts. *More often*,

the cysts are multiple and **small**. (J.A. del Regato, Diagnosis, treatment and prognosis, in: L.V. Ackerman (ed.) *Cancer* (1970))

Vagueness and its paradoxes have been discussed since antiquity. A famous example is the *Sorites* paradox which runs as follows. A man with a thick head of hair is not bald. If he looses one hair, he is not bald. If he looses another hair, he is still not bald. *Etc.* But then we seem forced to say that he is not bald even when he has lost all his hair.

A possible solution is that baldness is a matter of degree, not an all-or-none matter, and that pulling out one hair ever so slightly increases the degree of baldness. This idea is formalised by fuzzy logic, a particular brand of many-valued logic. The status of fuzzy logic is controversial. Allegedly, some modern devices such as autofocus cameras, rice cookers or washing machines owe their superior functioning to what is called 'fuzzy control', an implementation of fuzzy logic, and this has been heralded as the well-deserved success of a new way of thinking, *soft computing*. Others claim that there is no systematic reason for this success, and that it is mostly a matter of clever engineering. We shall now introduce fuzzy logic, together with some applications in fuzzy control and robotics, which will then give us a good starting point for a discussion of its merits and demerits.

2 Formal aspects of uncertainty and vagueness

Many-valued logic is a relative of first order logic obtained when the algebra of truth values $\{0, 1\}$ is replaced by another structure. Examples are the finite fields $\mathbb{Z}_p$ (i.e the set $\{0, \ldots, p-1\}$ (where p is a prime number) together with the operations $+$ and $\cdot$ taken modulo p) or the field of real numbers $\mathbb{R}$. Some applications require finite structures which are not fields. For instance in computability theory one sometimes considers a three-valued logic with truth values $\{t, f, u\}$ (for *true*, *false*, *undefined*) which have the structure of a partial order determined by the relations $f \leq u$, $u \leq t$.

A striking formal difference between probability and many-valued logic is that the latter is truth functional, whereas the former is not. E.g. in the absence of an independence assumption there is no way to compute $P(A \cap B)$ from $P(A)$ and $P(B)$, whereas by definition the truth value of $A \wedge B$ can be computed from the truth values of A and B. Two questions therefore immediately arise

1. is uncertainty truth functional?

2. is vagueness truth functional?

As regards question 1, a positive answer has been suggested by the proponents of the *certainty factors* model embodied in the expertsystem MYCIN. However, the justification given is mostly one of expediency: the assumption of truth functionality drastically reduces both time and space complexity of an expertsystem. Now consider the disadvantages of this assumption. Let $Deg(p)$ denote the (or my) degree of uncertainty of p, and suppose that $Deg(\neg p) = Deg(p)$. If Deg is truth functional, there is a fixed function F such that $Deg(p \wedge p) = F(Deg(p), Deg(p)) = F(Deg(p), Deg(\neg p)) = Deg(p \wedge \neg p)$. It is hard to come up with a situation where this is plausible.

The last example is also relevant to question 2: can we imagine a degree of vagueness Deg which has the property that, if $Deg(p) = Deg(\neg p)$, $Deg(p \wedge p) = Deg(p \wedge \neg p)$? This does not seem implausible if $Deg(p) = Deg(\neg p) = \frac{1}{2}$. More generally, meaning in natural language to a large extent seems to behave *compositionally*, i.e. the meaning of a complex expression can be taken to be a function of its constituents. Thus the meaning of the complex vague expression 'small and beautiful' is a function of the two vague expressions 'small' and 'beautiful'; and then it is but one step to argue that the degree of vagueness of 'small and beautiful' is some function of the degrees of vagueness of 'small' and of 'beautiful'. Then four important questions remain:

1. How should vagueness of atomic predicates be represented mathematically?
2. How does vagueness of formulas combine under logical operations?
3. How can we determine degrees of vagueness empirically?
4. What can one *do* in practice with degrees of vagueness?

In the following we shall treat these questions in turn. We will consider both fuzzy *sets* and fuzzy *logic*; we first comment on the relation between the two. For a good reference on fuzzy logic, see Hajek [2].

Fuzzy propositional logic has the same language as ordinary classical propositional logic, i.e. consisting of proposition letters $p, q, r, \ldots, p_0, \ldots$ (representing statements such as 'Laura has high fever') and the logical constants $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. The semantics of fuzzy propositional logic is rather different however; although it is usually taken to be a boundary condition that for the truthvalues $0, 1$ the logical constants behave classically. In classical propositional logic, $\rightarrow, \leftrightarrow$ are assumed to be definable from $\neg, \wedge, \vee$; this is not necessarily assumed here. Even talking about *the* semantics of fuzzy propositional logic is apt to be misleading; as we shall see there are many different possible semantics.
Similarly, fuzzy predicate logic has the same syntax as classical predicate logic, but differs in its semantics. Fortunately, in this case there appear to be no competing truth definitions for $\exists$ and $\forall$; the first is always defined by means of a supremum, the second by means of an infimum.

Logic is about validity of arguments, and fuzzy logic differs from many-valued logic in the peculiar type of inference rules it has proposed. We shall return to this matter below.

As to fuzzy sets, these are generalisations of standard sets (for which membership is a yes/no affair) to sets which allow degrees of membership. Operations on fuzzy sets A, B include union ($A \cup B$), intersection ($A \cap B$), complement ($\overline{A}$), but also for example analogues of operations on relations such as relational composition and taking the converse relation. Furthermore, an important concept here is that of *linguistic variable*, a kind of fuzzy variable whose values are themselves fuzzy sets. For example, 'age' can be taken to be a linguistic variable when its values are of the form {very young, young, middle-aged, old, very old}, instead of positive integers. This example shows where the name 'linguistic variable' derives from: the values are taken to be (the fuzzy sets denoted by) linguistic expressions. However, this is just accidental; the important thing is that the values are fuzzy sets. Linguistic variables are the most important ingredients of fuzzy control; they figure in control rules such as

> `IF` pressure is *high* `THEN` opening valve is *large*,

where the linguistic variables involved are 'pressure' and '(the size of the) opening (of the) valve'.

3 Fuzzy sets: mathematical representation

In standard set theory a sharp boundary exists between entities which belong to a particular set and entities which do not. Therefore it is not obvious how one should represent in standard set theory for example the set *high-fever* of patients with a high fever. The point at which one calls a fever 'high' depends on the context and is usually not completely determined. Each choice for a particular representation within standard set theory of *high-fever*, say for example $\{x : \text{Patient}(x) \wedge \text{Temp}(x) > 38.5\}$, is somewhat arbitrary and does no justice to the fact that a patient with a temperature of 40.5 degrees could be said to belong to *high-fever* with a higher degree than a patient with a fever of 38.6. In short, the extension of *high-fever* is *vague*. In the mid-sixties Lotfi Zadeh developed *fuzzy set theory* especially for representing sets with a vague extension. The following definition gives one particular mathematical representation; for motivation, recall that a standard set A may be represented as a *characteristic function* $\chi_A : V \longrightarrow \{0, 1\}$ (where V is the 'universe of all sets') such that $\chi_A(x) = 1$ iff $x \in A$.

Definition 3.1 *Let V be the universe, i.e., the (standard) set of the entities which are considered. A* fuzzy set *A is represented by a function $\mu_A : V \longrightarrow [0, 1]$. μ_A is* *called the* membership function, *or* compatibility function, *of A and $\mu_A(x)$ is called the* grade of membership *of x w.r.t. A. $\mu_A(x)$ can be considered to be a measure of the certainty with which one can say that x is an element of A. $\mu_A(x)$ is also called the* degree of truth *of the proposition that x is an element of A. The set $\{x : \mu_A(x) > 0\}$ is called the* support *of A. If the range of μ_A is $\{0, 1\}$, then A is called a* crisp set.

Example 3.1 *Let* small *be the set small positive integers. Then one might define $\mu_{small}(1) = 1$, $\mu_{small}(2) = 1$, $\mu_{small}(3) = 0.9$, $\mu_{small}(4) = 0.9$, $\mu_{small}(5) = 0.8$, etc.*

As noted in section 2, membership degrees do not have to be taken from $[0, 1]$; it is sufficient that they form a particular kind of algebraic structure. $[0, 1]$ is the most popular concrete set of membership grades, though. In order to define a membership function μ it is often sufficient to specify a number of parameters and a general type to which μ belongs:

Example 3.2 *The function $\mu_{high\text{-}fever}$ might be defined as follows.*
$\mu_{high\text{-}fever}(x) = L(x; 38.5, 41)$, where

$$L(x; a, c) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{c-a} & \text{if } a \leq x \leq c \\ 1 & \text{if } x > c \end{cases}$$

Example 3.3 *An alternative definition of $\mu_{high\text{-}fever}$ is the following. $\mu_{high\text{-}fever}(x) = S(x; 38.5, 39.75, 41)$, where*

$$S(x; a, b, c) = \begin{cases} 0 & \text{if } x < a \\ 2\left[\frac{x-a}{c-a}\right]^2 & \text{if } a \leq x \leq b \\ 1 - 2\left[\frac{c-x}{c-a}\right]^2 & \text{if } b \leq x \leq c \\ 1 & \text{if } x > c \end{cases}$$

The parameter b in $S(x; a, b, c)$ is chosen equal to $\frac{a+c}{2}$ and is in fact superfluous.

Both L and S have the property that they take a value of 0.5 in $\frac{a+c}{2}$, "half way"' from a to c. In addition, S is continuously differentiable, which according to Zadeh makes S the most appropriate membership function for increasing notions; like 'high', 'large', 'old', etc. Other important categories are the decreasing notions ('low', 'small', 'young', etc.) and the approximating notions ('about 100 meters', 'circa 30 years', 'near Utrecht', etc.). It is not hard to guess which function is considered to be the most appropriate for decreasing notions. Approximating notions are represented by the following function:

$$\pi(x;a,b) = \begin{cases} S(x; b-a, b-\frac{a}{2}, b) & \text{if } x \leq b \\ 1 - S(x; b, b+\frac{a}{2}, a+b) & \text{if } x \geq b \end{cases}$$

Although "neat" function like S or π are convenient representations of membership functions, it is by no means required that a fuzzy set is represented by such a "neat" membership function. In fact, there seems to be scant motivation for any particular type of membership function.

The meaning of *high-fever* as a fuzzy subset of for example $[35, 45]$ is given by the union of all expressions $\mu_{\text{high-fever}}(x)/x$, where $x \in [35, 45]$ and where $\mu_{\text{high-fever}}(x)/x$ means that the membership grade of x w.r.t. *high-fever* equals $\mu_{\text{high-fever}}(x)$. This union is sometimes written in the following form:

$$\text{high-fever} = \int_{35}^{45} \mu_{\text{high-fever}}(x)/x$$

N.B. The integral sign is just a notation for the union! If the support of a set A is finite, then the meaning of A can be given by means of an explicit enumeration of $\mu_A(x)/x$ for every x in the support of A, and instead of the integral sign, it is then customary to use '+.'

3.1 Linguistic variables

Now that we have fuzzy sets, we may also construct functions which have fuzzy sets as values (even when their arguments are crisp). Above we have already seen examples of such functions, for instance the function *age* which assigns to each integer n, $0 \leq n \leq 120$, a value in the set $\{$*very-young*, *young*, *young-adult*, *middle-aged*, *old*, *very-old*$\}$. Each of these labels represents a fuzzy subset of the interval [0,120].

Definition 3.2 *A* linguistic variable *is a function whose domain is a crisp set, and whose values are fuzzy sets.*

The name derives from the circumstance that the values can often be labelled by natural language expressions, but this feature is not essential. This notion of function was introduced to formalise a type of expression often used in fuzzy reasoning, such as 'the temperature is high' or 'muscle loss is moderate'; one then writes 'X is A', where X is the relevant linguistic variable, and A a fuzzy subset.

4 Where do the numbers come from?

So far our investigation has been mostly formal: we have defined membership functions and have given some examples, but we haven't said how membership functions should be determined in practice. Even when one chooses the standard membership functions provided by examples 3.2 and 3.3, then one still has to determine the values of the parameters. One might even plausibly argue that fuzzy logic tries to make a vague concept like 'high fever' too precise by insisting on a *numerical* representation.

Let us first discuss the relatively simple case of assigning generalised truth values (i.e. not just 0 or 1) to a proposition. Here one should sharply distinguish between *degree of belief* and *degree of vagueness* : one can have a degree of belief in the proposition

> the next toss with this coin will yield heads,

but strictly speaking not in the proposition

> this patient has high fever.

This becomes clear when one considers for example the justification of degrees of belief in terms of betting behaviour: one can only bet on the *truth* (or falsity) of a proposition, whereas 'this patient has high fever' is not simply true or false. Only when one has (artificially) assigned to 'high fever' a precise meaning, can a degree of belief be associated with this proposition. This shows that it is far from easy to combine uncertainty and vagueness in one formalism, even though, for example, putting medical knowledge into an expertsystem seems to require this.

There have been attempts to reduce generalised truth values to probabilities in the guise of frequencies by means of the following strategem: ask a large number of people what they think of the truth value of 'this patient has high fever' and take the average. However, even if this would be reasonable for atomic sentences, it cannot work for compound sentences, since there is no reason to assume that frequencies behave truthfunctionally.

In practice, for example in fuzzy control, the problem often takes a different form, because there the values yielded by measurement devices are initially taken to be precise. As we shall see, the heart of a fuzzy controller is a system of `IF...THEN` rules, whose antecedents and consequents contain linguistic variables. Somewhat surprisingly, the practical problem then takes the following form. First, a measured value has to be *fuzzified*, i.e. it is asssociated to a set of membership degrees of values of a linguistic variable. For example, the precise value '45' for *age* can be mapped on the set $\{0, 0, 0.2, 1, 0.2, 0\}$, corresponding to the fuzzy sets {*very-young*, *young*, *young-adult*, *middle-aged*, *old*, *very-old*}. These membership degrees of the measured value are then used for computing a set of membership degrees for some control variable (such as the width of the opening of a valve, etc.), using rules for `IF...THEN` that will be discussed later. Lastly, there follows the stage of *defuzzyfication*, converting the set of membership degrees into a single value. Various formulas have been proposed for this stage, but they are not our present concern. We thus see that in fuzzy control the practical problem boils down to the following: find a principled way to define a linguistic variable, i.e. to associate a membership function to a linguistic description. It is not impossible that there is a mathematical answer to this question: fuzzy controllers can be viewed as discretisations of differential equations, and it could be that one can prove that some membership functions are better suited for this purpose than others.

Expertsystems with fuzzy rules seem to be in a much worse position in this respect, at least when the expert has to assign a degree of truth to statements such as 'muscle loss is severe' on the basis of his/her own estimate, and not on the basis of, say, electromyography findings. Note that the procedure in case of a fuzzy controller is objective when the linguistic variables are given, i.e. there can be no dispute about the proper fuzzification of a measured value; whereas experts may strongly disagree about the degree of truth to be assigned to 'muscle loss is severe'.

In robotics applications of fuzzy logic one often has to give, not a single degree of truth, but an entire membership function. An example would be the function which, for a certain physical object, assigns a degree of truth to a location, representing the 'possibility' that the object is present at that location. This type of problem is further studied in section 8. We shall see that one can say something about the shape of those functions due to an unexpected connection with Dempster-Shafer theory; however, one generally has much less information about these membership functions than about probability distributions, where the underlying model often suggests a distribution coming from a wellknown family, so that only a few parameters have to be fixed.
In addition, it will become clear that there is a serious problem in interpreting predictions made on the basis of fuzzy logic. In section 8 we will be interested in self-localisation for a mobile robot. If the self-localisation function assigns 0 to a particular location, this should imply that the robot is not there; hence the meaning of this is reasonably straightforward. However, the function may assign 1 to several locations, not necessarily contiguous–unlike probabilities, these 'possibilities' need not sum to 1. To say that all these locations are 'fully possible' does not seem to elucidate much. The situation becomes worse when a certain location gets assigned a number α strictly between 0 and 1; does this really mean more than saying that the location cannot be discounted?

In sum, the justification of degrees of truth or degrees of membership is a weak point of fuzzy logic. Unlike the situation in probability theory, there is no principled way to obtain these; in fact, if one compares fuzzy logic to probability theory, than what fuzzy logic at present lacks is a counterpart of statistics.

5 Combining fuzzy sets

5.1 Boolean operations

One could define union and intersection on fuzzy sets by positing:

Definition 5.1 *Let A and B be fuzzy sets. The membership functions of $A \cap B$, $A \cup B$, and $\overline{A}$ are defined as follows:*

1. $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$
2. $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$
3. $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$

A well-known objection against these rules is that $A \cap \overline{A}$ (in standard set theory equal to $\emptyset$) has only an empty support in case $\forall x(\mu_A(x) = 0 \vee \mu_A(x) = 1)$. Analogously, $\mu_{A\cup\overline{A}}(x) = 1$ iff $\mu_A(x) = 0 \vee \mu_A(x) = 1$. In addition, the choice for the minimum function seems arbitrary. Some alternative rules have been proposed, but those of definition 5.1 are the most popular. Notice that the rules do satisfy the minimal requirement that standard set theory can be considered to be the special case of fuzzy set theory in which all sets are crisp.

A principled approach to the choice of a semantics would run as follows cf. Paris [4], chapter 5. One lays down general mathematical properties one would like a connective or set theoretic operation to have, and one then determines which possibilities are consistent with those properties. We give some examples of desirable properties. First for negation (or complement): its interpretation, also denoted $\neg$, is now considered as a function $\neg : [0,1] \longrightarrow [0,1]$:

N1 $\neg 0 = 1,\ \neg 1 = 0$

N2 $\neg$ is decreasing

N3 $\neg\neg x = x$ for $x \in [0,1]$

Principles N1-3 fairly constrain $\neg$; one can show

Theorem 5.1 *If $\neg$ satisfies N1–3, then $([0,1], \neg, \leq)$ is isomorphic to $([0,1], 1-x, \leq)$.*

Let's do the same trick for conjunction (or intersection); $\wedge$ should be a function $[0,1] \times [0,1] \longrightarrow [0,1]$ such that

C1 $0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1$

C2 $\wedge$ is continuous

C3 $\wedge$ is increasing (not necessarily strictly) in each coordinate

C4 $\wedge$ is associative.

A conjunction satisfying C1-4 is called a *t-norm*; one can show that any t-norm is also commutative. In a sense there are just three t-norms, as the following theorem shows.

Theorem 5.2 *Suppose $\wedge$ satisfies C1-4.*

1. *If for all $x \in [0,1]$, $x \wedge x = x$, then $\wedge = min$.*
2. *If for some a, b, c such that $0 \leq a < c < b \leq 1$: $c \wedge c = a$, then $([a,b], \wedge, \leq)$ is isomorphic to $([0,1], max(0, x+y-1), \leq)$*
3. *Otherwise, $([a,b], \wedge, \leq)$ is isomorphic to $([0,1], \times, \leq)$.*

Observe that only the second possibility gives a 'decent' value to the set $A \cap \overline{A}$. For completeness' sake we add the analogous conditions for disjunction (or union): $\vee$ must be a function $[0,1] \times [0,1] \longrightarrow [0,1]$ satisfying

D1 $0 \vee 0 = 0,\ 1 \vee 0 = 1$

D2 $\vee$ is continuous

D3 $\vee$ is increasing (not necessarily strictly) in each coordinate

D4 $\vee$ is associative

Results on disjunction follow directly from results on conjunction: if $\vee$ satisfies D1–4, then $\wedge$ defined by $x \wedge y = 1 - ((1-x) \vee (1-y))$ satisfies C1–4. We leave the converse to the reader. Applying theorem 5.2 then yeilds precisely three choices for $\vee$ satisfying D1–4, of which the maximum is one; these choices will be collectively called t-conorms.

5.1.1 Modifiers

Next to the usual Boolean operations on sets fuzzy set theory allows *modifiers* or *hedges*:

Example 5.1 *Let $\mu_{\textit{high-fever}}(x) = S(x; 38.5, 39.75, 41)$. The meaning of* very-high-fever *is built up from the meaning of* high-fever *and the meaning of* very. *Here* very *can be considered to be a* concentration modifier, *which modifies $\mu_{\textit{high-fever}}$ into $\mu_{\textit{very-high-fever}}$, for example as follows:*

$$mu_{\textit{high-fever}} = (\mu_{\textit{high-fever}}(x))^2.$$

This particular choice, due to Zadeh, has some drawbacks: e.g. if $\mu_{\text{high-fever}}(x) = 1$, then also $\mu_{\text{very-high-fever}}(x) = 1$, which is precisely what one does not want to have.

Example 5.2 *An alternative interpretation of* very *is that of a* translation modifier*, such as,* $\mu_{\textit{very-high-fever}}(x) = \mu_{\textit{high-fever}}(x + 0.5) = S(x; 39, 40.25, 41.5)$.

5.2 Composition

Lastly, a very important way of combining fuzzy sets is *relational composition.* In classical predicate logic, the composition $S \circ R$ of relations $S \subseteq X \times Y$ and $R \subseteq Y \times Z$ is defined by

$$(x, z) \in S \circ R \text{ iff } \exists y \in Y((x, y) \in S \;\&\; (y, z) \in R).$$

If we interpret & by the minimum, and $\exists$ by the supremum, we obtain the following analogue for fuzzy sets.

Definition 5.2 *Let $A \subseteq X$, $B \subseteq X \times Y$, and $C \subseteq Y \times Z$.*

1. *$A \circ B$, the* composition *of A and B (w.r.t. X) is given by:*

$$\mu_{A \circ B}(y) = \sup_{x \in X}(\min(\mu_A(x), \mu_B(\langle x, y\rangle)))$$

2. *$B \circ C$ (w.r.t. Y) is given by:*

$$\mu_{B \circ C}(\langle x, z\rangle) = \sup_{y \in Y}(\min(\mu_B(\langle x, y\rangle), \mu_C(\langle y, z\rangle)))$$

Obviously this definition has variants in terms of other t-norms, but we can safely leave this to the reader.

6 Fuzzy Logic

We now consider this material from the standpoint of fuzzy logic: i.e. we have to give an interpretation of the connectives, and a definition of valid consequence.

Definition 6.1 *Let $\mathcal{L}$ be a propositional language. A* (real) valuation *is a function v which to each formula assigns a real number from the interval* $[0, 1]$.

Since it is a distinguishing feature of fuzzy logic that it is truth functional, a valuation is completely determined if we know $v(p)$ for proposition letters p and in addition how v interacts with the connectives. We have seen above that, given N3, there is little choice for the interpretation of $\neg$: $v(\neg p)$ must equal $1 - v(p)$. The interpretation of $\wedge$

should be given by a t-norm, and that of $\vee$ by a t-conorm; in fuzzy logic one takes for these minimum and maximum respectively. In classical propositional logic, implication can be defined in terms of negation and disjunction; the reader may wish to check that the analogous definition in fuzzy logic would be unreasonable. One therefore needs a separate definition, for which fuzzy logic takes the so-called Lukasiewicz implication.[1] We therefore obtain:

Definition 6.2 *Fuzzy propositional logic has the syntax of classical propositional logic, and semantics given by real valuations v satisfying*

$$\begin{array}{lcl} v(\neg A) & = & 1 - v(A) \\ v(A \wedge B) & = & \min(v(A), v(B)) \\ v(A \vee B) & = & \max(v(A), v(B)) \\ v(A \rightarrow B) & = & \min(1, 1 - v(A) + v(B)) \end{array}$$

Definition 6.3 *Fuzzy predicate logic has the syntax of classical predicate logic, and semantics given by first order models $\mathcal{M}$ and real valuations v such that v satisfies the above rules for the propositional connectives and in addition (where $\overline{d}$ is a sequence of elements from $\mathcal{M}$)*

$$\begin{array}{lcl} v(\exists x\varphi(x), \overline{d}) & = & \sup\{v(\varphi(a, \overline{d}) \mid a \in \mathcal{M}\} \\ v(\forall x\varphi(x), \overline{d}) & = & \inf\{v(\varphi(a, \overline{d}) \mid a \in \mathcal{M}\} \end{array}$$ [2]

The next item on the agenda is the definition of valid consequence in fuzzy (propositional or predicate) logic, $\varphi_1, \ldots, \varphi_n \models_F \psi$.

Definition 6.4 *We say that ψ is a valid fuzzy consequence of $\varphi 1, \ldots, \varphi_n$, $\varphi_1, \ldots, \varphi_n \models_F \psi$, if for all first order models $\mathcal{M}$, all valuations v: $v(\varphi_1 \wedge, \ldots, \wedge \varphi_n) \leq v(\psi)$.*

One now could go on to ask questions about complete axiomatisations of fuzzy logic and its variants. However, this is a surprisingly difficult topic, and we prefer to let the matter rest. Instead, we shall add a few remarks on a special type of inference rule first proposed by Zadeh, which is of some importance in fuzzy control.

When describing a system in fuzzy terms, the basic building blocks are so-called *fuzzy implications*, that is, statements of the form '`IF` X is A, `THEN` Y is B', where the X, Y are linguistic variables, and A, B are fuzzy sets. The semantics of such a statement should be understood as follows. Since X is a linguistic variable, its domain D_X is a crisp set, and its values are fuzzy sets; likewise for Y. The idea is now

[1] Exercise 10.4 asks you to examine whether this choice is reasonable.

[2] Clearly $\forall$ is defined as a generalised $\wedge$. This raises the question whether there can be definitions of $\forall$ corresponding to other t-norms, such as $x \cdot y$. While in principle this can be done, in practice the definition of the quantifiers is kept constant.

to evaluate the generalised truth value of '$\texttt{IF}$ X is A, $\texttt{THEN}$ Y is B' for each pair $(x, y) \in D_X \times D_Y$; '$\texttt{IF}$ X is A, $\texttt{THEN}$ Y is B' thus corresponds to a fuzzy relation. However, we can only determine *which* fuzzy relation this will be if the meaning of $\texttt{IF...THEN}$ has been fixed. This is where the typical features of the fuzzy implication arise. One obvious possibility for $\texttt{IF...THEN}$ is implication in the sense of definition 6.2. This means that the value of the fuzzy relation corresponding to the $\texttt{IF...THEN}$ for arguments $(x, y) \in D_X \times D_Y$ can be computed from the values of $\mu_A(x)$ and $\mu_B(y)$. Interestingly, however, in applications one often needs a 'fuzzier' notion of fuzzy implication, which is not truthfunctional.
If '$\texttt{IF}$ X is A, $\texttt{THEN}$ Y is B' is interpreted truthfunctionally, then its truthvalue is less than 1 if $\mu_A(x) > \mu_B(y)$. In practice, one often wants to use $\texttt{IF...THEN}$ rules which allow exceptions; but then $\texttt{IF...THEN}$ can no longer be interpreted truthfunctionally. One can think of such rules as the analogues of probabilistic statements such as $P(Y \text{ is } B | X \text{ is } A) \sim 1$. For the counterexamples we then have $P(Y \text{ is not } B,\ X \text{ is } A) \sim 0$; we may loosely think of such an event as not occurring.
Fuzzy logic tries to incorporate this construction into its own framework by introducing a relation R on $D_X \times D_Y$ which isolates the events which are considered possible. $\texttt{IF...THEN}$ is now interpreted by means of this relation, i.e. we get for $(x, y) \in D_X \times D_Y$,

$$(\texttt{IF } X \text{ is } A, \texttt{ THEN } Y \text{ is } B)(x, y) = A(x) \wedge R(x, y) \rightarrow B(y).$$

Given R, write $(\mathcal{M}, R) \models (\texttt{IF } X \text{ is } A, \texttt{ THEN } Y \text{ is } B)$ to mean that in $\mathcal{M}$, for all instantiations of x, y: $v(A(x) \wedge R(x, y)) \leq v(B(y))$. Using the fact that $\forall x(A(x) \wedge R(x, y) \rightarrow B(y))$ is equivalent to $\exists x(A(x) \wedge R(x, y)) \rightarrow B(y)$, we then have

Lemma 6.1 *$(\mathcal{M}, R) \models$ ($\texttt{IF}$ X is A, $\texttt{THEN}$ Y is B) if for all $y \in D_Y$,*

$$\mu_B(y) \geq \sup_{x \in D_X} \min(\mu_A(x), \mu_R(x, y)).$$

The reader will recognise relational composition (cf. definition 5.2) in this definition. We will restate the lemma just given in a form that will be useful later. In terms of fuzzy predicate logic, the condition in the definition is equivalent to the truth of the statement

$$\forall y(\exists x(A(x) \wedge R(x, y)) \rightarrow B(y)).$$

Call this latter statement *Comp(A,B,R)*, then we have

Theorem 6.1 *Fuzzy predicate logic derives*[3]

$$\text{Comp(A,B,R)} \longrightarrow (A(X) \wedge R(X, Y) \rightarrow B(Y)).$$

[3]When the correct implication has been chosen.

Informally, if *Comp(A,B,R)* is true, i.e. if μ_B can be computed in a certain way from μ_A and μ_R, then from 'X *is* A' and '(X,Y) *is* R' we may infer 'Y *is* B'; this is what takes the place of *modus ponens* in fuzzy logic.

Here is an important special case, with `IF...THEN` taken to mean ordinary implication. Let A, A^*, B, B^* denote fuzzy sets, and define $Comp_{MP}$ to mean $\forall y(B^*(y) \leftrightarrow \exists x(A^*(x) \wedge (A(x) \rightarrow B(y)))$. Then one can prove

$$Comp_{MP} \wedge A^*(X) \wedge (A(X) \rightarrow B(Y))) \rightarrow B^*(Y).$$

Thus, if $Comp_{MP}$ has truth value 1, A^* has truth value r and $A(X) \rightarrow B(Y)$ has truth value s, then $B^*(Y)$ has truth value at least $\min(r,s)$. This inference rule is applicable in situations where the observation ('X *is* A^*') does not match the antecedent of the `IF...THEN` rule; e.g. if the rule says something about '`IF` X is *low*, `THEN` ...', and we know that X takes value *very-low*.

The important point to remember is that a statement of the form '`IF` X is A, `THEN` Y is B' has no unique semantics, but derives its meaning from a relation R. In practical applications, both in expertsystems and in fuzzy control, one often has the following situation: a system can be described at an intuitive level by a number of `IF...THEN` rules between linguistic variables; from these rules one then tries to determine an underlying relation R, which then represents the 'dynamics' or 'causal structure' of the system.

As a preparation for the discussion of an application of fuzzy logic in robotics, we now briefly study another interpretation of the membership function (cf. Paris [4], chapter 5).

7 Possibility Theory

The membership degree of fuzzy set theory can also be interpreted as a *degree of possibility*. For example, suppose patient P is known to have a high fever. In other words, the temperature of P is in the support of the fuzzy set *high-fever*. Then the degree of possibility of a particular x to be the temperature of P can be said to be equal to $\mu_{\text{high-fever}}(x)$.

The notions of probability and possibility are clearly different: the degree of possibility of 42 is for example equal to $\mu_{\text{high-fever}}(42) \approx 1$, whereas the probability that someone with high fever has a temperature of (exactly) 42 degrees is rather small. Hence improbable events can very well be quite possible. Of course, impossible events are improbable. Formally, possibility measures can be characterised as follows.

Definition 7.1 *Let Ω be a sample space. A function $\Pi : 2^\Omega \longrightarrow [0,1]$ is called a* possibility measure *iff it satisfies the following conditions.*

$$\Pi(\emptyset) = 0, \;\; \Pi(\Omega) = 1 \tag{1}$$

$$\textit{For all families}\{A_i\} \subseteq 2^\Omega, \Pi(\bigcup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i). \tag{2}$$

Notice that, in particular, $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$; but there is no such simple relation for $A \cap B$. The dual of a possibility measure Π is called a *necessity measure* N: $N(A) = 1 - \Pi(\overline{A})$.

For example, starting from a fuzzy membership function $\mu : \Omega \longrightarrow [0,1]$, one can define the measure Π as follows: $\Pi(A) = \sup_{x \in A} \mu(x)$. If one assumes that at least once μ has the value 1, then the resulting measure is a possibility measure.

Possibility theory is formally related to (a very special case of) DS theory. Recall that a *mass function* on a sample space Ω is a function $m : 2^\Omega \longrightarrow [0,1]$ such that $m(\emptyset) = 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$. In the present context, an appealing interpretation of mass functions arises when we assume given a space of sensors Θ together with a mapping $\Gamma : \Theta \longrightarrow 2^\Omega$ such that $\Gamma(\theta)$ is the imprecise outcome, i.e. nonempty *set of* outcomes, of sensor θ. If we also assume given a distribution of reliability weights over Θ whose sum equals 1, then Γ induces a mass function on 2^Ω. Call a mass function *consonant* if the elements of the core of m are nested, i.e. if $m(A) > 0$ and $m(B) > 0$, then $A \subseteq B$ or $B \subseteq A$. In terms of sensors, this means that the outcomes of any pair of sensors with nonzero reliability are comparable. It turns out that consonant mass functions are strongly related to possibility distributions. Recall that a plausibility function Pl is induced by a mass function m by means of the condition

$$Pl(A) \;=\; \sum_{B \cap A \neq \emptyset} m(B).$$

We then have

Theorem 7.1 *If the mass function m is consonant, then its plausibility function is a possibility measure. Conversely, given a possibility function Π one can find a unique mass function m such that the plausibility function Pl induced by m equals Π.*

Thus, if Π is a possibility measure, we may interpret $\Pi(\omega)$ as the total reliability of the sensors for which outcome ω is possible.

A slightly different interpretation arises when we recall that a mass function is determined by a family of probability distributions. We may then write the degree of possibility of A,

$$\Pi(A) \;=\; \sup\{P(A) \mid P \in \mathcal{F} \text{ a probability measure}\},$$

where $\mathcal{F}$ is the family of probability distributions determined by the mass function m, which is in turn given by the theorem 7.1.

8 Fuzzy logic in robotics

As an application of the preceding theoretical material, we illustrate how fuzzy logic can be used for self-localisation of a mobile robot. Clearly, when planning its future actions, a robot needs to know where it is at present. What's more, it needs to know this in order to interpret the data yielded by its sensors. In general it is not possible for a robot to determine its position by keeping track of where it is going (so-called 'dead-reckoning'); this is because there is too much noise in the effectors (such as wheel slippage). Typically, therefore, interpreting sensor data proceeds by matching perceptual clues with certain items on a stored map of the environment; but this can succeed only when the robot knows its approximate location on that map.[4] Indeed, perception and self-localisation mutually constrain each other in the following cycle

1. the sensors try to detect features in the environment; however, due to sensor noise identification of features is not foolproof, and unique identification is usually unattainable

2. the robot tries to match the (various possibilities for) features with items on the map, and then computes a self-location (or set of self-locations) which would explain the observed features

3. the more constrained self-localisation is used to obtain a better interpretation of the next round of sensor-data.

We now present a simple algorithm for self-localisation based on fuzzy logic, which is due to Saffiotti and Wesley [5]. Our presentation will differ slightly from the original in its emphasis on logic; the algorithm becomes very perspicuous when one writes a specification of what is needed in predicate logic.

Definition 8.1 *A* global location space *is a vector space G of the form $\mathbb{R}\times\mathbb{R}\times[0,2\pi]$, where the third coordinate denotes a direction with respect to the x-axis (i.e. the first coordinate); for example, the direction of motion of the robot.*
An approximate map *on G is a structure $\langle M, \mathtt{Type}, Pos\rangle$, where M is a non-empty set of names of objects;* `Type` *associates to each object an object type (e.g. door, wall, corridor, ...), and Pos is a binary relation on $M \times G$ such that for each $m \in M$, $g \in G$, the truth value of $Pos(m, g)$ denotes the possibility of m being present at g.*[5]

[4] Here and in the following we assume a fairly traditional view of AI, which works with representations. The opposed view, reactive robotics as pioneered by Brooks [1], holds that building up a representation is unnecessary and that all one needs is frequent perceptual sampling of the environment.

[5] If $Pos(m, g)$ is supplied by the designer of the robot, it can be assumed to be crisp; the generality of the definition is needed for the case where the robot has to build the map of the environment all

Here and in the following, we may interpret 'possibility' in the sense given in the previous section, i.e. as an upper bound of probabilities from a given family, as in theorem 7.1. In particular possibility 0 will mean probability 0 for any of the probability measures from the family. Thus, if the robot has possibility 0 to be at a certain location, we may feel confident in asserting that it is not there.

Definition 8.2 *The* space of local locations *is a subspace L of $\mathbb{R} \times \mathbb{R} \times [0, 2\pi]$. The origin of L is taken to be on the robot, such that the robot moves in the positive x-direction. Again, the third coordinate is added because it is sometimes useful to represent an object, such as a door, by means of a point and an orientation $\theta \in [0, 2\pi]$ (with respect to the robot's direction of motion).*
The crisp current locational hypothesis *is a vector $h \in G$ which represents the position of the origin of L in G. Given such an h, each location in L can be mapped onto a location in G by a coordinate transformation $T(h, \cdot) : L \longrightarrow G$:*

$$T\left(\left(\begin{array}{c} h_x \\ h_y \\ h_\theta \end{array}\right), \left(\begin{array}{c} l_x \\ l_y \\ l_\theta \end{array}\right)\right) = \left(\begin{array}{ccc} cos(h_\theta) & -sin(h_\theta) & 0 \\ sin(h_\theta) & cos(h_\theta) & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c} l_x \\ l_y \\ l_\theta \end{array}\right) + \left(\begin{array}{c} h_x \\ h_y \\ h_\theta \end{array}\right).$$

More generally, the current locational hypothesis *is a unary predicate H on G such that the truth value of $H(g)$ represents the possibility that the origin of L is located at g.*

The robot has stored an approximate map on G, and on the basis of its sensor data builds up a map of the environment in terms of coordinates from L. Due to sensor noise, the locations of the perceived objects on the map L will be fuzzy, and it may even be unclear to what type the perceived object belongs. Self-localisation consists in matching L with G, such that the expected local position of each object m, determined by means of a self-localisation hypothesis H, is (very nearly) equal to the observed local position. This motivates the following stipulations

Definition 8.3 Expected local position in L of m given H *is the following binary relation $EXP_H(m, l)$, defined by*

$$EXP_H(m, l) \leftrightarrow \exists g(g \in G \wedge H(g) \wedge Pos(m, T(g, l))).$$

Definition 8.4 Feat(p, t) *is a binary predicate whose truth value measures the degree of possibility that (observed) feature p is of type t.* Perc(p, l) *is a binary predicate the truth value of which represents the degree of possibility that (observed) feature p is at location $l \in L$.*

by itself. However, in that case the form of the functions $Pos(m, g)$ cannot be constrained a priori, and this may lead to computational problems.

Both sensing and the locational hypothesis yield approximate positions in L for an object m. If p is the feature sensed, then p is similar to m to the degree that p is of the type of m, and the degree that the expected position of m (given the current locational hypothesis) and p overlap. Both these aspects are taken into account in the definition of a similarity predicate:

Definition 8.5 *The degree of similarity of an observed feature p and a map object m, on the basis of a current locational hypothesis H is given by a binary predicate* $\mathrm{Sim}_H(p, m)$ *satisfying*

$$Sim_H(p,m) \;\leftrightarrow\; \exists l(l \in L \wedge EXP_H(m,l) \wedge Perc(p,l)) \wedge Feat(p, \mathtt{Type}(m)).$$

The next task is to build *localisers*, i.e. hypotheses about the robot's (fuzzy) location in G. We first define a ternary auxiliary predicate $Loc(p, m, g)$, whose truth value represents the degree of possibility that the robot is at $g \in G$, by measuring the overlap of p and the position in L of m, given g. It seems reasonable to assume that, the larger the locational similarity of m and p given g, the more plausible g is as a location of the robot (in G).[6] This predicate does not depend on the current locational hypothesis.

Definition 8.6

$$Loc(p,m,g) \;\leftrightarrow\; \exists l(l \in L \wedge Pos(m, T(g,l)) \wedge Perc(p,l)).$$

We now want to update the current locational hypothesis H by means of the location estimate just defined. H gave us the degree of similarity $Sim_H(p, m)$ between m and p; now if for instance this degree is small, then the locations g which would lead to large overlap between m and p are not plausible as locations for the robot. Furthermore, the observed p may be explained by several different m. We thus construct the binary predicate $Loc(p, g)$ by 'averaging' over m, taking into account that there may be no m to which p is very similar; this is done by means of an implication with antecedent $\exists m(m \in M \wedge Sim_H(p, m))$.

Definition 8.7

$$Loc(p,g) \;\leftrightarrow\; [\exists m(m \in M \wedge Sim_H(p,m)) \rightarrow \exists m(m \in M \wedge Loc(p,m,g) \wedge Sim_H(p,m))].$$

Note that the effect of the implication is that if there is no m matching p, then $Loc(p, g)$ is vacuous in the sense that every location gets possibility 1.

The preceding definition of the updated H took into consideration only one observed feature p. Now suppose the set of observed features at time t is P_t; if $H = H_t$, then $Loc(p, g)$ is implicitly relativised to t.

[6] In probability theory this argument would be suspect; it smacks of the base rate fallacy, i.e. assuming $P(A \mid B) = P(B \mid A)$.

Definition 8.8 *Given the current locational hypothesis $H = H_t$, the updated hypothesis H_{t+1} is obtained by*

$$H_{t+1}(g) \;\leftrightarrow\; \bigwedge_{p \in P_t} Loc(p, g).$$

If additional information $I(g)$ is available, for instance information obtained by dead-reckoning, then the previous definition can be modified by adding a $\ldots \wedge I(g)$ clause.

What we have done so far is to give the specification of the self-localisation problem in terms of predicate logic. Although we indicated informally what the formulas are supposed to mean, what we did so far is pure syntax. On the semantic side, this corresponds to a computation whose inputs are

1. functions $G \longrightarrow [0, 1]$ interpreting $Pos(m, \cdot)$
2. functions $L \longrightarrow [0, 1]$ interpreting $Perc(p, \cdot)$
3. functions $L \longrightarrow [0, 1]$ interpreting $Feat(p, \cdot)$

and whose output is a function $H : G \longrightarrow [0, 1]$. To perform the actual computation, we need to know the shapes of the input functions, and the computational meaning of the logical constants used. For instance, as regards the latter, we have seen that there are various possible interpretations for the logical constants in a many-valued setting. The choice of fuzzy logic would be $x \vee y = \min(x, y)$, $x \wedge y = \max(x, y)$, $x \rightarrow y = \min(1, 1 - x + y), \exists = \sup$. The choices for input functions and interpretation of logical constants should be determined by internal consistency and by what can be termed *performance criteria.*

In the implementation used by Saffiotti and Wesley [5], the self-localisation function is evaluated at each control cycle of the robot. This leads to the following desiderata on performance:

1. the 'informativeness' of H should be nondecreasing with time,
2. if H assigns zero possibility to a location, then the robot should not be there,
3. the support of H should converge fairly rapidly to an area little larger than that occupied by the robot.

None of these desiderata has been formulated very precisely, and this indeed seems hardly possible. Part of the problem is that it is hard to say how to evaluate values of H strictly between 0 and 1 in practice. If these values were probabilities, we would be in a slightly better position, because in that case we could try to find some kind of frequency model. However, possibilities are not probabilities, although a case can be made for stating that possibility 0 means probability 0. Thus perhaps condition 1 could be glossed as saying that the support of H_t should be nonincreasing (and preferably decreasing) with time t. Fortunately, the algorithm is such that, on any interpretation of the logical constants, the support of H_t is nonincreasing with t.

However, this then leads to the problem of what the relevance of the precise shape of the fuzzy sets is; after all, there are very many fuzzy sets with the same support. For example, one may consider on the one hand, the familiar 'hat' shape, on the other hand a function which assigns possibility 1 to each element of the support (i.e. an ordinary set). In an intuitive sense, the latter function has less information than the former, but it is not clear whether the difference matters in practice.

Looking at this problem from another angle: how are the shapes of the fuzzy sets determined in this particular application? 'Flakey', the robot on which the algorithm outlined above was implemented, only has sonar sensors, and this influences the reliability of the data obtained; in general the uncertainty in distance is much less than the uncertainty in angle. Take for instance the function $Perc(p,l)$, which measures the possibility that feature p is located at l. Consider first the case that p is a wall. As the robot moves along the wall, it will get fairly accurate readings for the distance of the wall, but the length component will be largely undetermined. Hence $Perc(p,l)$ will be such that it is precise with respect to the y and θ coordinates, but vague with respect to the x coordinate. One may then try to transfer the reliability of the sonars to their readings, as indicated in section 7. It is clear that desideratum 2 must already be satisfied at this stage, if it is to be satisfied later. To continue with the example, if p is a door, the sonars can fairly reliably sense the position of the door in the wall, but they may have extreme difficulty determining its orientation. This suggests that $Perc(p,l)$ has the following form: an almost precise pair (x,y) which fixes the point where the door is attached to the wall, and a largely undetermined value for the orientation θ; in an extreme case $Perc(p,l)$ could be the product of a very narrow 'hat' for (x,y) and the function $\theta \equiv 1$. However, the sensor data will not determine the shape of the fuzzy set uniquely; the situation is different from that in probability theory where one often may determine the general shape of the probability distribution on theoretical grounds (e.g. as being of normal or exponential type), so that only a few parameters have to be determined empirically.

9 Fuzzy control

Autofocus cameras, washing machines, microwaves, but also older feats of engineering such as steam engines, contain control loops in order to ensure stable behaviour of some sort. For instance, in an autofocus camera, the controller first determines what is to be figure (i.e. subject), and what ground, and then by means of a cyclic feedback

process adjusts the lens so as to get the figure sharply in focus. In the case of a steam engine driving a steam turbine which is desired to have constant a number of revolutions, the pressure in the boiler must be adjusted continuously depending on the deviation between desired and actual number of revolutions of the turbine.
Fuzzy logic has been applied in all these cases, and in this section we intend to discuss some of the principles behind these applications. The general situation is as follows. We have a system S, yielding output $x(t)$, where t is the time variable. The output signal is compared to the desired output $w(t)$. If $x(t)$ differs significantly from $w(t)$, then a corrective signal $y = y(x, w, t)$ is supplied to S with the purpose of bringing $x(t)$ more in line with $w(t)$. The task is then to determine the most appropriate y. This depends on the amount of deviation one is willing to tolerate, the speed with which correction has to be effected, the cost of executing a corrective action, and, last but not least, the desire to obtain a stable system.
Mathematically, the problem takes the following form: the system is governed by a differential equation

$$\frac{dx}{dt} = h(x, t, y), \quad x(0) = c, \tag{3}$$

where y is to be chosen so as to satisfy the aforementioned goals. E.g. if $\sigma(x - w)$ is some way of measuring the deviation of x from w (say the quadratic difference), then we are interested in a solution of 3 which minimises $\int_0^T \sigma(x - w)dt$. This is not an easy problem[7], and fuzzy logic has been applied here in part to ease the computational difficulties: it is clear that one does not want to equip household appliances with elaborate computing hardware.

As an example we shall treat the case of a steam engine driving a steam turbine, following Mamdami and Assilian [3]. We concentrate on the control loop consisting of the revolution counter on the turbine and the valve determining the amount of steam let into the turbine.
Typically, a fuzzy controller consists of a number of `IF...THEN` rules, stated in terms of several linguistic variables. The variables of interest in our application are

- *VC*, the required change in the valve opening
- *DR*, the deviation in the number of revolutions
- *CDR*, the *change* in *DR* with respect to the last measurement

The `IF...THEN` rules are then all of the form

`IF` *DR* is A `AND` *CDR* is B, `THEN` *VC* is C,

[7]It belongs to the calculus of variations; computational techniques to solve it are sometimes available in what is known as dynamic programming.

Rule	*DR*	*CDR*	*VC*
1	*LN*	$\neg$ (*LN* $\vee$ *MN*)	*LP*
2	*MN*	*LP* $\vee$ *MP* $\vee$ *SP*	*SP*
3	*SN*	*LP* $\vee$ *MP*	*SP*
4	*0*$^-$	*LP*	*SP*
5	*0*$^-$ $\vee$ *0*$^+$	*SP* $\vee$ *SN* $\vee$ *0*	*0*
6	*0*$^+$	*LP*	*SN*
7	*SP*	*LP* $\vee$ *MP*	*SN*
8	*MP*	*LP* $\vee$ *MP* $\vee$ *SP*	*SN*
9	*LP*	$\neg$(*LN* $\vee$ *MN*)	*LN*

where the A, B, C are fuzzy sets, often identified by means of linguistic descriptions. For example, the descriptions could be *LP* (*large-positive*), *MP* (*medium-positive*), *SP* (*small-positive*), *O* (*zero*), *O*$^+$ (second order positive difference from 0), *O*$^-$ (second order negative difference from 0), *SN* (*small-negative*), *MN* (*medium-negative*) and *LN* (*large-negative*). Each of these descriptions is then associated to a fuzzy set over the possible values of the valve-opening or the number of revolutions; we need not inquire how this is done exactly. The `IF...THEN` rules can then be represented by means of a table such as the following:
A few remarks on this fuzzy controller are in order. Firstly, the rules only have an exact meaning when the meaning of `IF...THEN` has been fixed, i.e. the underlying relation R. Mamdami and Assilian choose for this the relation

$$MAMD(x, y) \;=\; \bigvee_{i \leq 9} A_i(x) \wedge B_i(y),$$

where A_i is the $\wedge$ of the values of *DR* and *CDR* in the ith row, and B_i is the value of *VC* in the ith row. The meaning of *MAMD* is that it singles out the pairs (x, y) which are compatible according to the dynamics of the system. Its use in derivations is the following. Given a unary predicate A^*, define a unary B^* from A^* by forcing *Comp* to be true, i.e. we put

$$\forall y(\exists x(A^*(x) \wedge MAMD(x, y)) \rightarrow B^*(y)).$$

Theorem 9.1 *In fuzzy logic, the following formula is valid*[8]

$$\left[\bigwedge_i (\texttt{IF}\ X\ \textit{is}\ A_i\ \texttt{THEN}\ Y\ \textit{is}\ B_i)\ \wedge \bigvee_i A_i(X)\right] \rightarrow (A^*(X) \rightarrow B^*(Y)).$$

[8] I.e. when the correct implication is chosen!

That is, if the truthvalue of the antecedent (the part between [ ]) is 1, then the truthvalue of $A^*(X)$ is less than or equal to that of $B^*(Y)$. The relation $MAMD(x, y)$ is thus necessary to allow the derivation of new `IF...THEN` rules, featuring predicates which do not occur in the original rules. This is important considering the third point below.

Secondly, on an intuitive level the rules are plausible, and could be written down without much detailed knowledge of the process involved: if the target value is a constant, the rules tell one what to do given information about the number of revolutions and the first order derivative of the number of revolutions. We thus have a qualitative analysis of equation 3 above. For instance the fourth rule says that if the number of revolutions is only a little less than the desired value, and the number of revolutions has been increasing rapidly, then one should effect a *small* change in the direction opposite to that of the deviation (to prevent overshoot). The very unspecificity of this controller might of course also be a drawback. This is related to the point that it is in general not possible to *prove* that a system governed by a fuzzy controller is stable.

Thirdly, not all combinations of (fuzzy) values occur in the antecedent; for example the situation where DR takes value LP and CDR takes value LN is not considered explicitly. `IF...THEN` rules for this situation can be derived using the rule of inference given by theorem 9.1.

Fourthly, it should be recalled that these rules come together with procedures for fuzzification and defuzzification of measured values, in order to supply the link with experience.

Fuzzification is fairly straightforward. One measures a precise outcome for a linguistic variable X, which then determines a number for each fuzzy value A of X. Using the `IF...THEN` rules, these numbers then determine degrees of membership for fuzzy values B of a linguistic variable Y. These numbers have somehow to be combined to yield a single number which can be fed into an effector; this is defuzzification. Various strategies have been proposed, all of them apparently rather arbitrary, so we shall not discuss them here.

10 Exercises

Exercise 10.1 Let $v(A)$ and $v(B) \in \{0, 1\}$. Show that the truth values of $A \wedge B$, $A \vee B$, $\neg A$ en $A \rightarrow B$ in fuzzy logic are those of classical two-valued proposition logic.

Exercise 10.2 Show that in fuzzy logic $\neg A \vee B$ and $A \rightarrow B$ are not equivalent.

Exercise 10.3 For each pair of t-norms, find a principle on $\wedge$ which is valid according to one member of the pair, but not according to the other.

Exercise 10.4 Although in many-valued logic $\rightarrow$ is not definable in terms of the other connectives, it is related to them, in the following sense. Let $\leq$ be the order in $[0,1]$, and $*$ any t-norm. Then (the interpretation of) the implication $\rightarrow$ should satisfy

$$\text{for all } x, y, z \in [0,1]\ (x * z) \ \leq\ y \text{ iff } z \ \leq\ x \rightarrow y.$$

Why is this a necessary property for an implication? Using this property, compute for each t-norm the corresponding implication.

Exercise 10.5 Can one find a consistent interpretation of the certainty factor formalism inside many-valued logic? If so give the interpretation; if not, state what goes wrong and how one might remedy this.

Exercise 10.6 Verify that the following predicate-logical principles are valid for any t-norm:

$\forall 1$ $\forall x(\varphi(x) \rightarrow \varphi(s))$ (s a term substitutable for x in φ)

$\exists 1$ $\varphi(s) \rightarrow \exists x\varphi(x))$ (s a term substitutable for x in φ)

$\forall 2$ $\forall x(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \forall x\varphi)$ (x not free in ψ)

$\exists 2$ $\forall x(\varphi \rightarrow \psi) \rightarrow (\exists x\varphi \rightarrow \psi)$ (x not free in ψ)

$\forall 3$ $\forall x(\varphi \vee \psi) \rightarrow (\forall x\varphi \vee \psi)$ (x not free in ψ).

Exercise 10.7 Assuming the above five principles, prove for the case that ψ does not contain x free

1. $\exists x(\psi \rightarrow \varphi) \rightarrow (\psi \rightarrow \exists x\varphi)$
2. $\exists x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \psi)$

Are the converse implications derivable in standard predicate logic? Are they valid when $\wedge$ is interpreted as product? Are they valid when $\wedge$ is interpreted as min? Are they valid when the t-norm is instead taken to be $x * y \ \equiv\ \max(0, x + y - 1)$?

Exercise 10.8 Show that for a necessity measure N one has $N(A \cap B) = \min(N(A), N(B))$.

Exercise 10.9 Show that if the capacity F is both a possibility measure and a probability measure, then F can only assign the values 0 and 1.

Exercise 10.10 Show that a consonant mass function induces a possibility function.

Exercise 10.11 Saffiotti and Wesley used fuzzy logic to interpret their formulas. Discuss what would change if instead of interpreting $\wedge$ by min, one would use the product for this purpose.

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